NUMERICAL SOLUTION OF SINGULARLY PERTURBED MULTIPLE TURNING POINT PROBLEMS

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NUMERICAL SOLUTION OF SINGULARLY PERTURBED MULTIPLE TURNING POINT PROBLEMS

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Abstract

The work contained in this thesis is mainly based on the author's research on the numerical analysis of a class of singularly perturbed differential equation with a multiple boundary turning points on a closed domain. The theory of singular perturbations is not a settled direction in mathematics and the process of its improvement is a dramatic one. With the escalated advancement of science and technology, numerous practical problems, for example, the mathematical boundary layer theory or approximation of solutions of different problems portrayed by differential equations involving large or small parameters, become progressively unpredictable, and, in this manner, in their examination, it is normal to utilize asymptotic methods. In certain problems, the perturbations are operated across a very narrow region over which the dependent variable experiences extremely fast changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the way that the small parameter multiplies the highest derivative. As a result, they are usually alluded to as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum mechanics, and Stokes lines and surfaces in mathematics. Boundary turning point problems, on the other hand, arise naturally in geophysics and in modeling thermal boundary layers in laminar flow. In particular, singularly perturbed turning point problems received much attention in the literature due to the complexity involved in finding uniformly valid asymptotic expansions, unlike non-turning problems.

The research aims to design a numerically consistent scheme for the singularly perturbed multiple turning point problems that is efficient and robust than the existing methods. The objectives of this research are to conduct a thorough investigation into the properties of singularly perturbed multiple turning point problems, to design and structure the numerical scheme based on spline collocation, to determine the convergence of the constructed scheme and to examine the solution profiles of the test problems using MATLAB software and compare the efficiency and accuracy with the existing solutions.

Since the multiple turning point problem has a boundary layer in the vicinity of $x = 0$, therefore, a fitted piecewise uniform Shishkin mesh (S-mesh) was introduced to circumvent the oscillations as $\epsilon \to 0$ that discretises $\bar{\Omega} = [0, 1]$ with $N = \{2^m, m \geq 2\}$ mesh elements. A minimum principle, stability estimate, and bounds on the solution and its derivatives are established. B-spline collocation on a fitted S-mesh is used to obtain the approximate solution to the multiple turning point problems. Previously obtained bounds are applied in the convergence analysis of the illustrated scheme. Furthermore, it is proved that the proposed method is uniformly convergent with respect to the singular perturbation parameter ϵ. Some relevant numerical examples are illustrated to verify the theoretical aspects computationally and the results are compared with other existing methods to show that the proposed method provides more accurate solutions.

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List of Publications

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Contents

1 Introduction

1.1 Differential Equation

A differential equation is a mathematical equation that associates some function with its derivatives [142]. In applications, the functions generally represent physical quantities, the derivatives indicate their rates of change, and the differential equation defines a relationship between the two. As such relations are extremely general, differential equations play a key role in numerous disciplines like engineering, physics, economics, and biology.

In pure mathematics, differential equations are considered from alternate points of view, generally caught with their answers to the arrangement of capacities that fulfill the condition. Only the simplest differential equations are solvable by explicit formulas; nonetheless, a few properties of arrangements of a given differential condition might be resolved without finding their precise structure. On the off chance that a closed-form expression for the solution isn't accessible, the solution might be numerically approximated utilizing computers. The hypothesis of dynamical frameworks puts an accentuation on the subjective investigation of frameworks depicted by differential conditions, while numerous numerical strategies have been created to decide arrangements with a given level of precision.

Differential conditions previously appeared with the development of calculus by Newton and Leibniz. In Chapter 2 of his 1671 work, 'The Method of Fluxions and Infinite Series', Isaac Newton [102] recorded three types of differential conditions:

$$\tfrac{dy}{dx} = f(x), \quad \tfrac{dy}{dx} = f(x,y), \quad \text{and} \quad x_1 \tfrac{\partial y}{\partial x_1} + x_2 \tfrac{\partial y}{\partial x_2} = y.$$

In all these cases, y is an unknown function of x (or $x_1$and x_2), and f is a given function. He solves these examples and others using infinite series and examines the non-uniqueness of these solutions.

In 1695, Jacob Bernoulli [15] proposed the Bernoulli differential equation given of the form

$$y' + P(x)y = Q(x)y^n.$$

In the following year, Leibniz found its solution by simplifying it [46]. In the year 1750, the Euler-Lagrange equation [120] was developed regarding their investigations of the tautochrone problem. This is the issue of determining a curve on which a weighted particle will fall to a fixed point in a fixed measure of time, independent of the starting point. Lagrange tackled this issue in 1755 and sent the answer to Euler. Both further built up Lagrange's technique and applied it to mechanics, which prompted the definition of Lagrangian mechanics. In 1822, Fourier [39] published his work on heat flow in 'The Analytic Theory of Heat', where he put together his reasoning on Newton's law of cooling, namely, that the flow of heat between two adjacent molecules is proportional to the incredibly little contrast of their temperatures. Contained in this book was Fourier's proposal of his heat equation for conductive diffusion of heat. This partial differential equation is now educated to each understudy of mathematical physics.

1.2 Singularly Perturbed Boundary Value Problems

The theory of singular perturbations is not a regulated direction in mathematics and the way in which it is enhanced is dramatic. As science and technology advance, many practical problems, such as mathematical boundary layer theory or approximation of solutions of various problems modelled by differential equations involving large or small parameters, become increasingly unpredictable, and asymptotic methods are commonly used to investigate them. In any case, the asymptotic analysis of differential operators has a created hypothesis in the case of regular perturbations, when the perturbations convey a subordinate character as for the unperturbed operator. In certain problems, the perturbations are operated across a very narrow region over which the dependent variable experiences extremely fast changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the way that the small parameter multiplies the highest derivative. As a result, they are usually alluded to as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum mechanics, and Stokes lines and surfaces in mathematics.

Towards the end of the nineteenth century, the study of fluid mechanics was veering in two unrelated directions: theoretical hydrodynamics and hydraulics. The former developed from Euler conditions for inviscid flows and accomplished a high level of fulfillment. Unfortunately, the results attained by utilizing this classical science remained in glaring logical inconsistency to test results. The celebrated d'Alembert's paradox is an illustrative example. This stimulated the researchers to build up their own exact study of hydraulics which depended basically on a large number of experimental data. In his seminal paper on "Fluid motion with very small friction", Prandtl demonstrated that the flow about a body can be treated by partitioning it into two regions: a very thin layer in proximity to the body (which he called boundary layer) where frictional effects are prominent, and the remaining outside region. Based on this speculation, Prandtl emphasized the significance of viscous flows without digging into the mathematical complexities involved. This boundary layer theory later turned into the establishment stone for modern fluid dynamics. Accordingly, the introduction of singular perturbations occurred at the Third International Congress of Mathematicians in Heidelberg in 1904. Prandtl's [109] seven-page report was contained in the proceedings of the conference. Nonetheless, the term "singular perturbations" was first utilized in the work of Friedrichs and Wasow [40], a paper which pursued a productive New York University seminar on nonlinear vibrations. The solution of singular perturbation problems commonly contains layers. Even though Prandtl presented the terminology "boundary layer" in this conference, it received a lot more prominent sweeping statements in the considerable work of Wasow [131].

Numerical analysis and asymptotic analysis are two principal approaches towards solving singular perturbation problems. Since the objectives and the problem classes are rather different, there has not been a lot of connections between these techniques. The numerical analysis attempts to give quantitative information about a specific problem, while asymptotic analysis attempts to pick up knowledge into the qualitative behaviour of a family of problems and only semi-quantitative information about any specific member of the family. Numerical methods are planned for expansive classes of problems and are planned to minimize demands upon the problem solver. Asymptotic methods treat similarly confined classes of problems and require the problem solver to make them comprehend the behaviour of the solution expected. Since the mid-1960s, the area of singular perturbations has steadily developed to the point where the subject is generally part of graduate understudies tackling problems of applied mathematics and numerous fields of engineering. Various great reading materials has shown up in this area which either managed with an asymptotic approach or with numerical ones. This list is quite extensive, but a noteworthy few are mentioned below: Bellman [12], Kaplun [64], Van Dyke [122], Hemker and Miller [53], Hughes [56], Brauner et al. [17], Doolan et al. [26], O'Malley [107, 106], Morton [94], Verhulst [124, 125], Roos et al. [114], Axelsson et al. [9], Eckhaus [29, 30], Nayfeh [100, 101], Holmes [54], Kevorkian and Cole [67], Miller et al. [90], and Bender and Orszag [13].

A numerical method based on the asymptotic expansion technique and the reproducing kernel method (RKM) [44] was used for solving singularly perturbed turning point problems exhibiting an interior layer. A modification of the Shishkin discretisation mesh [130] was designed for the numerical solution of one-dimensional singularly perturbed reaction-diffusion problems. The modification consists of a slightly different choice of the transition points between the fine and coarse parts of the mesh. The change did not affect the order of convergence of the numerical solution obtained by using the central finite-difference scheme. Berger et al. [14] derived bounds for the derivatives of the solution of a singularly perturbed turning point problem. They used the modification of the El-Mistikawy Werle finite-difference scheme at the turning point that showed a uniform error estimate for the resulting method.

Kadalbajoo et al. [59] proposed a numerical scheme to solve singularly perturbed two-point boundary value problems with a turning point exhibiting twin boundary layers. The scheme comprises of a cubic spline collocation method on a uniform mesh using artificial viscosity, which leads to a tridiagonal linear system. Geetha and Tamilselvan [42] constructed a robust numerical method based on a finite-difference scheme on a Shishkin mesh for a class of convection-diffusion type turning point problems with Robin type boundary conditions. Mittal [93] proposed a numerical method to approximate the solution of the nonlinear parabolic partial differential equation with Neumann's boundary conditions. It is based on the collocation of cubic B-splines over finite elements so as to maintain the continuity of the dependent variable and its first two derivatives throughout the solution range. The method was applied to spatial variables and its derivatives, which produced a system of first-order ordinary differential equations.

Zhang et al. [141] considered a collocation method for the numerical solution of fourth-order partial integrodifferential equations. The scheme is based on the second-order backward differential formula in the time direction and the quintic B-spline method for the spatial derivatives. A one-dimensional reaction-diffusion-convection problem is numerically solved by a finite element method on two layer-adapted meshes, Duran-type mesh and Duran-Shishkin-type mesh, both defined by recursive formulae [18].

Kadalbajoo and Reddy [63] conducted a study of different asymptotic and numerical methods developed for the determination of the approximate solution of singular perturbation problems of various kinds. Kadalbajoo and Patidar [61, 62] broadened the work done by Kadalbajoo and Reddy. In this work, they considered one-dimensional problems only and discussed the work done on linear, nonlinear, semilinear and quasilinear problems.

The one-dimensional singular perturbation problems

$$-\epsilon y'' + p(x)y' + q(x)y = g(x), \quad y(a) = \alpha, \quad y(b) = \beta,$$

can be partitioned under the classes given in Table (1.2.1).

Table 1.2.1: Summary of a variety of linear problems.

Conditions of $p(x)$	Types of solutions
Case 1: $p(x) \neq 0$ on $a \leq x \leq b$: i) $p(x) < 0$ ii) $p(x) > 0$	 i) Boundary layer at $x = a$. ii) Boundary layer at $x = b$.
Case 2: $p(x) = 0$: i) $q(x) > 0$ ii) $q(x) < 0$	 i) Boundary layers at $x = a$ and $x = b$. ii) Rapidly oscillating solution.
Case 3: $p'(x) \neq q(x)$, $p(0) = 0$: i) $p'(x) < 0$ ii) $p'(x) > 0$	 i) No boundary layers, an interior layer at $x = a$. ii) Boundary layers at $x = a$ and $x = b$, no interior layer at $x = a$.

The advancement of small parameter methods prompted the productive use of boundary layer theory in different fields of applied mathematics, for example, fluid mechanics, fluid dynamics, elasticity, quantum mechanics, plasticity, chemical-reactor theory, aerodynamics, plasma dynamics, magneto-hydrodynamics, rarefied gas dynamics, oceanography, meteorology, diffraction theory, reaction-diffusion processes, non-equilibrium and radiating flows and different areas of the classes of fluid motion. In this, some singular perturbation models which emerge in various branches of applied sciences and engineering are listed below:

Consider the one-dimensional Schrodinger equation [25]

$$\epsilon \frac{d^2\psi_\epsilon}{dx^2} + (\lambda_\epsilon + V(x))\,\psi_\epsilon = 0, \quad \|\psi_\epsilon\| = 1$$

with $V(x)$ (potential) continuous and leading to $+\infty$ as $|x| \to \infty$ and with $\epsilon = h\frac{(2m)^{\frac{1}{2}}}{2\pi}$, h denoting Planck's constant and m is the mass. Associated with these ordinary differential equations, the eigenvalue problem in the Hilbert space $L^2(-\infty, +\infty)$ with the norm $\|u\| = \left(\int\limits_{-\infty}^{+\infty} u^2(x)\,dx\right)^{\frac{1}{2}}$ has a discrete spectrum with eigenvalues $\lambda_{\epsilon,1}$, $\lambda_{\epsilon,2}$, ..., $\lambda_{\epsilon,n}$, ... and eigenfunctions $\psi_{\epsilon,1}$, $\psi_{\epsilon,2}$, ..., $\psi_{\epsilon,n}$, ...

A time-independent Fokker-Planck equation [45] for a one-dimensional dynamical system with state-independent random perturbations are represented by the following equation:

$$\epsilon^2 \tfrac{d^2\varphi}{dx^2} + b(x)\tfrac{d\varphi}{dx} = 0, \quad 0 < \epsilon \ll 1, \quad x \in (0,1), \quad \varphi(0,\epsilon) = A, \quad \varphi(1,\epsilon) = B,$$

where $b(x)$ denotes a gradient field. Under the assumptions that $b'(x)$ are strictly negative throughout the interval $[0,1]$ and that $b(\gamma) = 0$ for some $0 < \gamma < 1$, the above problem becomes a resonant turning point problem.

A shock wave in a one-dimensional nozzle flow with the governing steady-state Navier-Stokes equations [8] gives

$$\epsilon A(x)uu'' - \left[\frac{1+\gamma}{2} - \epsilon A'(x)\right]uu' + \frac{u'}{u} + \frac{A'(x)}{A(x)}\left(1 - \frac{\gamma-1}{2}u^2\right) = 0, \quad 0 < x < 1,$$

where x is the normalized downstream distance from the throat, u is a normalized velocity, $A(x)$ is the area of the nozzle at x, e.g., $A(x) = 1 + x^2$, $\gamma = 1.4$ and ϵ is essentially the inverse Reynolds number, i.e. $\epsilon = 4.792 \times 10^{-8}$. The boundary conditions are $u(0) = 0.9129$ (supersonic flow in the throat) and $u(1) = 0.375$. For this boundary value problem an $O(\epsilon)$-wide shock develops, whose location depends on ϵ.

The swirling flow between two rotating, coaxial disks [8], located at $x = 0$ and $x = 1$ with the boundary value problem gives

$$\begin{aligned}
&\epsilon f'''' + ff''' + g' = 0, \quad 0 < x < 1,\\
&\epsilon g'' + fg' - f'g = 0,\\
&f(0) = f(1) = f'(0) = f'(1) = 0,\\
&g(0) = \Omega_0, \quad g(1) = \Omega_1,
\end{aligned}$$

where Ω_0 and Ω_1 are the angular velocities of the infinite disks, $|\Omega_0| + |\Omega_1| \neq 0$, and ϵ is a velocity parameter, $0 < \epsilon \ll 1$. For this boundary value problem, several solutions are possible. For instance, when $\Omega_1 = 1$, one can obtain different cases for different values of Ω_0. If $\Omega_0 < 0$ (with a special symmetry when $\Omega_0 = -1$), then the disks are counter-rotating; if $\Omega_0 = 0$ then one disk is at rest, while if $\Omega_0 > 1$ then the disks are co-rotating.

Consider an example from the theory of shells of revolution [8]. The ordinary differential equations are

$$\begin{aligned}
&\epsilon^2\left[\psi'' + \tfrac{1}{x}\psi' - \tfrac{1}{x^2}\psi\right] - \tfrac{1}{x}\phi\left(\phi_0 - \tfrac{1}{2}\phi\right) = 0,\\
&\epsilon^2\left[\phi'' + \tfrac{1}{x}\phi' - \tfrac{1}{x^2}\phi\right] - \tfrac{1}{x}\psi\left(\phi_0 - \phi\right) = 2\kappa p(x),
\end{aligned}$$

where ϕ is the meridional angle change of the deformed middle surface and ψ is a stress function. $\phi_0(x)$ is ϕ of the undeformed shell (for a spherical shell $\phi_0(x) = x$, also consider $\phi_0(x) = x^m, \quad m = 2, 3$), and

$$p(x) = x\left(1 - \gamma + \tfrac{\gamma}{2}x^2\right), \quad \gamma = 1.2, \quad \upsilon = 0.3, \quad \kappa = 1.$$

The boundary conditions are $\phi(0) = \psi(0) = 0, \quad \phi(1) = \psi'(1) - \upsilon\psi(1) = 0$. For this boundary value problem an interior layer (corresponding to dimpling) forms in a solution for ϕ. There is an additional boundary layer at $x = 1$ and more than one solution exists. The value $\epsilon = 10^{-4}$ (which gives a rather thin shell) yields a challenging numerical problem.

The mathematical model describing the motion of the sunflower [108] is given by the equation

$$\epsilon x''(t) + ax'(t) + b \sin x(t-\epsilon) = 0, \quad \epsilon > 0, \quad t \in [-\epsilon, 0],$$

with $x'(0)$ prescribed. $x(t)$ refers to the angle of the plant with the vertical, ϵ is a geotropic reaction, a and b are positive parameters obtained experimentally.

Consider the boundary layer flow of an electrically conducting incompressible fluid (with electric conductivity σ) over a continuously moving flat surface with B_0 an imposed, uniform magnetic field perpendicular to the surface. The boundary layer equation [121] for the flow field (in the nondimensionalized form) is then given by

$$\begin{aligned}
&\epsilon \tfrac{\partial^2 u}{\partial y^2} + \tfrac{\partial u}{\partial y} = \tfrac{\partial u}{\partial t} + Mu, \text{ for large } \mathrm{R}_0 \left(\epsilon = \tfrac{1}{R_0}\right), \\
&\tfrac{\partial^2 u}{\partial y^2} + \epsilon \tfrac{\partial u}{\partial y} = \epsilon \tfrac{\partial u}{\partial t} + \epsilon Mu, \text{ for small } R_0, \\
&u(0,t) = 1, \quad u(y,t) \to 0 \text{ as } y \to \infty, \quad t > 0, \\
&u(y,0) = 0, \quad 0 < y < \infty,
\end{aligned}$$

where $R_0 = \frac{V_0 L}{v}$, the suction Reynolds number, $M = \frac{\sigma B_0 L}{\rho V_0}$, the Hartmann number, $u = \frac{U}{U_w}$, $y = \frac{Y}{L}$, $t = \frac{\tau V_0}{L}$, L is the characteristic length (between the slit and the wind-up roll), V is the suction velocity at the plate, τ is the time and the X and Y axes are taken along and its perpendicular to the sheet, u is the velocity field.

The free motion of the undamped linear spring-mass system [107] with a very resistant spring and prescribed specific displacement at times $t = 0$ and $t = 1$ is governed by the two-point problem

$$\epsilon \ddot{x} + x = 0, \quad 0 \leq t \leq 1, \quad x(0) = 0, \quad x(1) = 1,$$

where ϵ^2 (the ratio of the mass to the spring constant) is small. For non-exceptional small positive values of ϵ, the exact solution oscillates rapidly, so no pointwise limit exists as $\epsilon \to 0$.

Assume an isothermal atmosphere [4], which is viscous and thermally conducting, possesses the upper half-space $z > 0$. Let's consider small oscillations about equilibrium which depend only on the time t and on the vertical coordinate z and let p, ρ, w and T be the perturbations in the pressure, density, vertical velocity and temperature respectively, and P_0, ρ_0, and T_0 refers to the equilibrium quantities. Then the linearised equations of motion are:

$$\rho_0 w_t + p_z + g\rho = \tfrac{4\mu w_{zz}}{3},$$
$$\rho_t + (\rho_0 w)_z = 0,$$
$$\rho_0 \left[c_v \left(T_t + qT\right) + gHw_z\right] = \kappa T_{zz},$$
$$P = R\left(\rho_0 T + T_0 \rho\right)$$

with prescribed boundary conditions, where μ refers to the dynamic velocity coefficient, κ is the thermal conductivity, c_v is the specific heat at constant volume and q is the Newton cooling which refers to the heat exchange and proportional to the temperature perturbation associated with the wave, R is the gas constant, g is the gravitational acceleration and $H = \frac{RT_0}{g}$ is the density scale height. It is useful to introduce the dimensionless parameter $\epsilon = \frac{\kappa}{\mu}$, $\epsilon \propto \frac{1}{P_r}$, where P_r is the Prandtl number, which measures the relative strength of the effect of the viscosity with respect to that of the thermal conductivity.

1.3 Turning Point Problems

Numerous phenomena in biology, chemistry, engineering, and physics can be portrayed by boundary value problems associated with different types of differential equations or systems. At whatever point a mathematical model is related to a phenomenon, the researchers by and large attempt to capture what is essential, retaining the important quantities and overlooking the insignificant ones which involve small parameters. The model that would be obtained by maintaining the small parameters is called the perturbed model, while the simplified model (the one that does not include the small parameters) is called the unperturbed (or reduced) model. For research purposes, the perturbed model can be supplanted by its unperturbed counterpart yet what is important is that its solution must be close enough to the solution of the corresponding perturbed model. This reality holds great in the case of regular perturbation but, in the case of singular perturbation, it is probably not going to hold. These singular perturbation problems with or without turning point(s) normally happen in numerous parts of applied mathematics, e.g., as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum mechanics and Stokes lines and surfaces in mathematics. In these sorts of problems, perturbations are usable over a very narrow region across which the dependent variable experiences fast change. These narrow regions every now and again append the boundaries or some interior point of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Therefore, these sorts of problems exhibit boundary and/or interior layers, that is, there are narrow regions where the solution changes quickly.

Singularly perturbed differential equations with turning point form a significant class of problems that are exceptionally challenging and even today there is a great deal to be investigated in this area. Additionally, problems where discontinuity in the data result in interior layers in the solution of the problem commonly occur during modeling of physical processes. Singular perturbation problems with turning points emerge as mathematical models for different physical phenomena. Among these, the problem with interior turning points represents the one-dimensional version of stationary convection-diffusion problems with a prevailing convective term and a speed field that changes its sign in the catch basin. Boundary turning point problems, on the other hand, arise in geophysics [48] and in the modeling of thermal boundary layers in the laminar flow [115]. The problem from [48] models heat flow and mass transport near an oceanic rise. It is a single boundary turning point problem because of the assumption that the velocity distribution is linear. On the off chance that one takes into consideration higher-order velocity distribution, at that point, it turns into multiple boundary turning point problems [115].

A typical linear turning point problem in one dimension is given by

$$\epsilon y'' + a(x)y' + b(x)y = 0, \quad x \in [x_1, x_2], \quad x_1 < 0, \quad x_2 > 0,$$

where $0 < \epsilon \leq 1$, $a(x)$ and $b(x)$ are sufficiently smooth. This issue has procured a lot of enthusiasm amongst mathematicians just as physicists because of the way that the arrangement shows some intriguing conduct, for example, boundary layer, interior layer and resonance phenomena. When $a(x)$ does not change the sign in the whole interval $[x_1, x_2]$, the solution is described by a boundary layer near one endpoint as $\epsilon \to 0$. When $a(x)$ has a simple zero, say at $x_0 = 0$ in $[x_1, x_2]$, the point x_0 is the so-called turning point and the problem is categorized as a turning point problem. In this circumstance, the solution behaviour depends upon the properties of the coefficient functions $a'(x)$ and $b(x)$ at the turning point $x_0 = 0$. Indeed, if it is assumed that, for α, β constants, $a(x) \sim \alpha x$ and $b(x) \sim \beta$ as $x \to 0$ the following cases arise:

1. If $\alpha > 0$, $\frac{\beta}{\alpha} \neq 1, 2, 3, ...$, an internal layer occurs near the turning point $x_0 = 0$.
2. If $\alpha < 0$, $\frac{\beta}{\alpha} \neq 0, -1, -2, ...$, then there are two boundary layers appearing at the two endpoints of the interval.
3. If $\alpha < 0$, $\frac{\beta}{\alpha} = 0, -1, -2, ...$, or if $\alpha > 0$, $\frac{\beta}{\alpha} = 1, 2, 3, ...$, the solution exhibits a very interesting phenomenon named as Ackerberg-O'Malley's [3] resonance phenomenon.

Another circumstance where the interior layer emerges would be the situation of singularly perturbed convection-diffusion-reaction problems based on smooth data. On the off chance that at least one coefficients such as the convection term, reaction term, source term or the boundary conditions are discontinuous, the solution of such type of problems exhibits strong or weak interior layers depending on the magnitude of the singular perturbation parameter and the nature of the coefficients.

Furthermore, some singular perturbation models with turning point(s) that simulate some real-world problems are listed below. It also discusses some models where the interior layer occurs due to discontinuity in the coefficients or non-smoothness of the data.

One-dimensional equation [13] which describes a quantum mechanical particle in a potential is given by

$$\left(-\epsilon^2\frac{d^2}{dx^2}+V(x)-E\right)y(x)=0,$$

where $V(x)$ and E is the potential and total energy of the particle respectively. If the equation $Q(x)=V(x)-E$, then $Q(x)$ vanishes at points where $V(x)=E$ and these are called turning points. The classical orbit of a particle in the potential $V(x)$ is confined to regions where $V(x)\leq E$. The particle moves until it reaches a point where $V(x)=E$ and then it stops, turns around and moves off in the opposite direction.

Consider the singularly perturbed boundary value problem [16]

$$\epsilon\ddot{x}+g(x)\dot{x}^q=0,\quad 0<t<T,\quad x(0)=A,\quad x(T)=B,$$

where $q=0$ or $1\leq q\leq 2$, ϵ refers to a fixed positive infinitesimal parameter, T, A and B are standard values, $g(x)$ is a smooth function and the interval $[A,B]$ contains zeros of $g(x)$. When $q=1$, the boundary value problems occur in many different applied contexts as in the study of exit time problems for stochastic differential equations [87] and when $q=0$, it occurs in reaction-diffusion and phase transition models [38], the study of contrast structures [20] and problems related to critical paths of Feynman integral.

Black-Scholes equation [77] models the financial data by means of

$$\tfrac{\partial C}{\partial t}+\tfrac{1}{2}\sigma^2S^2\tfrac{\partial^2 C}{\partial S^2}+rS\tfrac{\partial C}{\partial S}-rC=0,\quad (S,t)\in\Re^+\times[0,T),$$

with the final condition

$$C(S,T)=\max(S-E,0),\quad S\in\Re^+,$$

and the boundary conditions at $S=0$ and at $S=+\infty$

$$C(0,T)=0,\quad C(S,t)\to S,\ \text{for}\ S\to\infty,\quad t\in[0,T),$$

where $C=C(S,t)$ is a European call option, S and t are the current values of the underlying asset and time; σ, E, T and r refers to volatility, exercise price, expiry time and the risk-free interest rate respectively. This problem has a few singularities, for example, the unbounded domain, and the piecewise smooth initial function (its first-order derivative in x has a discontinuity of the first kind at the point $x = 0$).

In the stationary modeling of a semiconductor device [91], the model equations governing the static one-dimensional case are given by

$$\begin{aligned}
&\psi'' - \eta e^{\psi} + \rho e^{-\psi} = -D, \\
&\left(e^{\psi}\eta'\right)' - R\left(\psi, \eta, \rho\right) = 0, \\
&\left(e^{-\psi}\rho'\right)' - R\left(\psi, \eta, \rho\right) = 0,
\end{aligned}$$

on $\Omega = (0, 1)$ with appropriate boundary conditions at $x = 0$ and $x = 1$. ψ is the electrostatic potential, η and ρ are the electron and hole concentrations in the Slotboom variables, D refers to the doping function and R is the recombination/generation rate. The doping function D has a large jump at a point Ω called $p{-}n$ junction. The magnitude of the jump falls in the range from 10^{10} units to 10^{20} units. The solutions η and ρ have thin interior layers in the neighborhood of the $p - n$ junction because of this jump. The singular perturbation parameter λ is a function of $\eta e^{\psi} + \rho e^{-\psi}$.

The Allen-Cahn equation [6] arising in material sciences is given by

$$\begin{aligned}
&\epsilon^2 \Delta u + u - u^3 = 0 \text{ in } \Omega, \\
&\tfrac{\partial u}{\partial v} = 0 \text{ on } \partial\Omega,
\end{aligned}$$

where $u(x)$ is a continuous realisation of the phase in a material confined to the region Ω at the point x and v is the outer unit normal to $\partial\Omega$. For this problem, there are solutions that take value close to ± 1 except for narrow regions known as transition layers.

In this section, a survey of the various numerical and asymptotic techniques used by the researchers over some time to deal with the singularly perturbed turning point problems are given. Throughout the section, it is assumed that ϵ is a small parameter such that $0 < \epsilon \ll 1$.

1.3.1 Asymptotic Approach

Probably the most frequent approaches in the early days were the asymptotic approaches. They have information on the solution to the problems qualitatively (and semi-quantitatively).

Hanson and Russell [49] considered the system of the form

$$\epsilon^h \frac{dy}{dx} = A(x, \epsilon) y,$$

where $A(x,\epsilon)$ is a 2×2 matrix having uniform asymptotic expansion. $A(x,\epsilon) = \sum_{r=0}^{\infty} A_r(x)\epsilon^r$ as $\epsilon \to 0$ in $|\arg\epsilon| \leq \alpha_0$ and holomorphic for $\{(x,r) : |x| \leq \delta_0, 0 < |\epsilon| \leq \epsilon_0,\ |\arg\epsilon| \leq \alpha_0\}$. They proved that either the above problem can be reduced to one solvable by elementary methods or that the lead matrix $A(x,\epsilon)$ can be modified by some explicit variables to form $A_0(x) = \begin{bmatrix} 0 & x^k \\ x^{k+p} & 0 \end{bmatrix}$, where k and p are non-negative integers. They also showed that further simplifications can be obtained using transformations involving asymptotic sequence in the powers of ϵ for cases $k=0$, $h>0$, $p>0$, or if $h=1$, $k>0$, $p\geq 0$.

Dorr and Parter [27] examined the asymptotic behaviour of the solutions $u(t) = u(t,\epsilon)$ and $v(t) = v(t,\epsilon)$ to nonlinear boundary value problems of the form

$$
\begin{aligned}
&u'' = f(t,u,v),\ \ u(0) = u(1) = 0\ \ (0<t<1),\\
&\epsilon v'' + g(t,u,u')v' - c(t,u,u')v = 0,\ \ v(0) = v_0,\ \ v(1) = v_1\ \ (0<t<1),
\end{aligned}
\tag{1.3.1}
$$

where $0 \leq v_0 < v_1$, $c \geq 0$, f, g, c are continuous and $|f(t,u,v)| \leq f_0(t,v)$ for $t \in [0,1]$, $v \in [0,v_1]$. Some of the monotonic properties of the solutions were derived after proving the existence of solutions in $C^2[0,1]$ and convergence (as $\epsilon \to 0^+$) to some limiting functions $U(t)$, $V(t)$ of the family $u(t,\epsilon)$, $v(t,\epsilon)$. The "reduced equation" satisfies these limiting functions (in some sense). For the case where these problems exhibit a turning point phenomenon, the authors offered an asymptotic approach to estimate the solution of such types of problems. The problems with interior turning points as well as problems where the endpoints of the domain are turning points are considered and some very helpful findings on the asymptotic behaviour of the solution have been derived.

Ackerberg and O'Malley [3] presented the first systematic treatment of singularly perturbed boundary value problems with turning points. They considered the boundary value problem of the form

$$
\begin{aligned}
&\epsilon y'' + f(x;\epsilon)y' + g(x;\epsilon)y = 0\ \ (-a<x<b),\\
&y(-a) = A\ \text{ and }\ y(b) = B,
\end{aligned}
\tag{1.3.2}
$$

where A, B are real constants, a, $b>0$, $f(x;\epsilon)$ has a simple zero in $[-a,b]$ at $x=0$ and $f'(x;\epsilon) < 0$ throughout $[-a,b]$. The authors used the method of matched asymptotic expansion, the WKBJ method (Wentzel-Kramers-Brillouin-Jeffreys) and the asymptotic solutions of differential equations derived from Sibuya [117] to research the above problem. They defined the problems encountered by using matched asymptotic expansions when dealing with such problems and also discussed the need to rely on the method of WKBJ. They found out that paradoxical outcomes are obtained by the matched asymptotic expansion approach by formal analysis of a variety of interesting examples. In $(-a,b)$ with a boundary layer at $x=-a$ if $I>0$ and at $x=b$ if $I<0$, a nontrivial solution is expected, where $I = \int_{-a}^{b} f(x;0)dx$. When $I = 0$, the boundary layer appears at both endpoints. A turning point solution is valid for $x = O\left(\sqrt{\epsilon}\right)$ if $I \neq 0$, while a solution satisfying the reduced equation and one of the boundary conditions is obtained elsewhere in $(-a, b)$. The authors provided a good discussion of the problems involved in the study of this complex phenomenon and concluded that the method of matched asymptotic expansion is insufficient to provide a good approximation to solving such problems because it produces the following paradoxical results:

1. Boundary layers of the form $C_1 \exp\left(\frac{-k_1(x+a)}{\epsilon}\right)$ and $C_2 \exp\left(\frac{-k_2(b-x)}{\epsilon}\right)$ are considered to be possible at both endpoints, although only one endpoint has boundary layer if $I \neq 0$ where C_1, C_2 are arbitrary constants independent of x and ϵ.

2. It is not possible to achieve a particular asymptotic expansion as one of the constants in the expansion remains undetermined after all the matching is finished. They suggested a method based on WKBJ expansion to handle certain types of problems in order to address this discrepancy.

O'Malley [105] studied the asymptotic behaviour of the solution of the differential equation in the form

$$\epsilon y'' + 2xA(a,\epsilon)y' - A(x,\epsilon)B(x,\epsilon)y = 0, \quad -1 < x < 1,$$

as where y has arbitrary values independent of ϵ at $x = -1$ and $x = 1$, $A(x,\epsilon)$ and $B(x,\epsilon)$ are assumed to be real and holomorphic in $|x| \leq 1$ and possesses asymptotic expansions in the powers of ϵ as $\epsilon \to 0^+$. $A(x,\epsilon)$ is non zero throughout $[-1,1]$. The author used Lee's technique [75] to use asymptotic power series to transform the given differential equation into Weber's equation. Then, in terms of parabolic cylindrical functions, he obtained the desired asymptotic form of the solution above, via the Weber equation solution. He discovered that the solution depends decisively on the sign of $A(x,0)$ in the following ways:

1. If $A(x,0) > 0$ then for $B(0,0) \neq -2n$, $n = 1,2,3,...$, the limit of the solution is a function that satisfies both the boundary conditions and solves the differential equation to the left as well as to the right of the turning point. In general, there is a discontinuity at the turning point, while if $B(0,0) = -2n$, $n = 1,2,3,...$, the solution becomes exponentially large for $-1 < x < 1$.

2. If $A(x,0) < 0$ then $B(0,0) \neq 2m$, $m = 0,1,2,...$, tends to zero in $-1 < x < 1$ for the solution, while if $B(0,0) = 2m$, the limiting solution becomes non-trivial in $-1 < x < 1$.

The results of Ackerberg and O'Malley [3] were expanded by Watts [132] to study the structure of the solution of the problem (1.3.2) for the case where $g(x;\epsilon)$ is continuously differentiable and $f(x;\epsilon)$ is twice differentiable. He assumed that $f(0;\epsilon)=0$, $f'(0;\epsilon)=-1$, $f(x;\epsilon)<0$ for $0<x\leq 1$ and $g(0;\epsilon)=b$. To obtain the approximate solution by a suitable construction of an integral equation, the author used two variable procedures and proved that this solution is an approximation to some exact solution of the given problem. The nature of the solution is compared based on the value of b in the following ways:

1. If b is neither a positive integer nor zero, then the solution gives boundary layer at each point with an exponentially small value in the interior of the domain.
2. If b is a positive integer, it is not possible to find approximate solutions to two-point boundary value problems, such as those discussed by Ackerberg and O'Malley [3].

The author found that the approximate solution obtained by him was not valid for intervals extending to ∞, and the two variable procedure needs to be continued to higher-order terms to find the correct approximate solution. In addition, he considered the exact solution of the problem (1.3.2) with $f(x;\epsilon)=-x$ and $g(x;\epsilon)=b+x$ and found that resonance does not occur when $b=n$, where n is a non-negative integer and only occurs if $b=n-\epsilon$ which contradicts the results in [3].

The problem (1.3.2) for the case $l=0,1,2,...$ was examined by Kreiss and Parter [70], and completed the results of Ackerberg and O'Malley [3]. For $x\neq 0$, they assumed $xf(x,\epsilon)<0$ with a single simple zero in $(-a,b)$ where $a,b>0$ for the function $f: f(0,\epsilon)=0$. The basic estimates of the regularity of the solution $y(x,\epsilon)$ were established and showed that

$$\begin{aligned}
&\|y\|_{-a,b}\leq K_0\left[\|y\|_{-\delta,\delta}+|A|+|B|\right],\\
&\|D_x^j y\|_{-a+\delta,b-\delta}\leq K_0\left[\|y\|_{-\delta,\delta}+\epsilon\left(|A|+|B|\right)\right],\quad j=0,1,...,k+1,
\end{aligned}$$

for $0<\epsilon\leq\epsilon_0$, $\epsilon_0=\epsilon_0(\delta)$, $k_0(\delta)$ be a constant and $0<\delta<\min(a,b)$. They demonstrated that the global behaviour of $y(x,\epsilon)$ is dependent upon its local behaviour and indicated that if (y,ϵ_n) is a family of solutions of the problem (1.3.2) which are then uniformly bounded, there is a series that converges uniformly on each subinterval $[a+\delta,b-\delta]$ along with their derivatives of order $1,2,...,k$. The researchers analyse the nature and position of the boundary layer and discovered that if $\{\epsilon_n\}$ satisfies condition B (a sequence $\epsilon_n\to 0^+$ satisfies condition B if there exist a constant $K_1>0$ and functions $\{w_1(x,\epsilon_n)\}$ that satisfy (1.3.2) and $w_1(-a,\epsilon_n)=1$, $\|w_1'\|_{-a,b}\leq K_1$) and $I=\int_a^b f(x,0)dx$, then $\bar{\epsilon}>0$ exists such that:

1. for all A, B, and $\epsilon_n\leq\bar{\epsilon}$, $y(x,\epsilon_n)$ has a unique solution to the problem (1.3.2).
2. a reduced equation solution $u(x)$ exists such that $\lim\limits_{n\to\infty}\|y(x,\epsilon_n)-u\|_{-a+\delta,b-\delta}=0$.
3. if $I>0$, then $u(b)=B$ and near $x=b$ there is no boundary layer.
4. if $I<0$, then $u(a)=A$ and at $x=a$ there is no boundary layer.

5. if $I = 0$, then $u(x) = \lambda\hat{u}(x)$, where $\lambda = \frac{Af(-a,0)-Bf(b,0)\hat{u}(b)}{f(-a,0)-f(b,0)|\hat{u}(b)|^2}$ and $\hat{u}$ is the solution of the reduced equation in the neighborhood of the origin.

In case (3) and (4), their results matched with the Ackerberg and O'Malley's [3] result, but in case (5), the different value of λ was yielded. For the exceptional case in which $l = \frac{-g(0,0)}{f(0,0)}$ is a non-negative integer, the authors discussed some results and examples.

The singularly perturbed boundary value problems was considered by Matkowsky [86] for the differential equation of the form

$$\begin{aligned} &\epsilon y'' + f(x;\epsilon)y' + g(x;\epsilon)y = h(x;\epsilon), \quad -a < x < b, \\ &y(-a;\epsilon) = \alpha(\epsilon), \quad y(b;\epsilon) = \beta(\epsilon), \end{aligned} \tag{1.3.3}$$

where $a, b > 0$ and f changes sign at one or more points in the interval under consideration. For such problems, the author proposed conditions for resonance occurrence and derived uniformly valid asymptotic expansions. If $w(x;\epsilon) = \sum_{j=0}^{\infty} w^j(x)\epsilon^j$ is the outer expansion where $w^0(x)$ follows the reduced equation and at $x = b\,(x = -a)$ if $f(x;\epsilon)$ has positive (negative) boundary conditions, then functions $w^j(x)\,(j \geq 1)$ are defined by inserting w in (1.3.3) and separately equating the coefficients of each power of ϵ to zero. He demonstrated that resonance occurs when all the functions $w^j(x)$ in the outer expansion are x-limited functions in the domain, including $x = 0$. In a series of examples, in the case where $l = \frac{-g(0,0)}{f'(0,0)}$ is a non-negative integer, the author solved the problem of the triviality of the limiting solution within the interval. For the problems with $f' > 0$ on intervals containing the origin, the method proposed in the present work was found to be inapplicable. By adding the variable $\xi = \frac{x}{\sqrt{\epsilon}}$ and studying the resulting differential equation in ξ, the author studied the behaviour of the solution in the neighborhood of $x = 0$ whose solution is then matched as $\xi \to \pm\infty$ to the outer solution in terms of parabolic cylindrical functions $D_n(\xi)$. Also, the author linked the resonance phenomenon to a problem of eigenvalue and showed that the particular values that cause resonance are the eigenvalues of some problem of homogeneous boundary value.

Howes [55] provided sufficient conditions for the existence of the solution of

$$\epsilon y'' = f(t, y, y', \epsilon),$$

prescribing $y(-1)$, $y(1)$ and $\frac{\partial f}{\partial y'}$ with $t = 0$. He assumed that, when f is continuous, the solution to the reduced problem is well behaved and showed that the solution might have the following depending on the behaviour of the turning point: (i) the boundary layer at $t = -1$ or $t = 1$, (ii) boundary layers at both the end points and (iii) the interior transition layer. He generated explicit boundary layer solutions and the estimates were obtained for the transition layers of the solution.

In order to create a priori estimates for the solution of the problem (1.3.2), Abrahamsson [2] generalized the earlier work of Kreiss and Parter [70]. The results that he has obtained depend on the value of $a'(x)$ as follows:

Case 1. $a' < 0$.

1. If $c < 0$, $b < 0$, then

$$\|y\|_{\infty}^{2} \leq K \left(\max_{|x|\leq\delta} |f|^{2} + \|f\|_{[\delta\leq|x|\leq 1]}^{2} + \alpha^{2} + \beta^{2} \right)$$

   where $\|.\|$ is L_2 norm and ϵ, δ is small but positive whereas for $b > 0$, $l = \frac{b(0)}{|a'(0)|}$, $l = 0, 1, 2, ...$, the corresponding estimate involves $|f_k^{\infty}|^2$ near zero for $k = 0$ and $k = l + 1$. By using some examples, the author further demonstrated that the necessity of differentiability on f is only required in a neighborhood of the turning point and can be eliminated if $|f| = O\left(\sqrt{\epsilon}\right)$, $l > 0$ for $|x| \leq \delta$.

2. If $c = 0$, in this case if $b(0) > 0$ and $l \neq 1, 3, 5, ...$, then

$$\|y\|_{\infty}^{2} \leq K \left(\max\nolimits_{0\leq x\leq\delta} |f(x)|^{2} + \max\nolimits_{0\leq x\leq\delta} \left|f^{(2k)}(x)\right| + \|f\|^{2} + \beta^{2} \right),$$

   provided $\alpha = 0$ and $f(x) = x^{2m} g(x)$, $2m \geq 2k$, $2k-2 \leq l \leq 2k$, whereas if $b(0) > 0$, then there is a maximum principle for $0 \leq x \leq \delta$ and a priori estimates hold good.

Case 2. $a' > 0$.

In this case, if $l \neq 1, 2, 3, ...$, then solutions are defined beyond the neighborhood of the turning point by convergence to the solution of the reduced equation, while the solution and higher order derivatives are unbounded in the neighborhood of $x = 0$.

The case of resonance for the problem

$$\begin{aligned} &\epsilon \tfrac{d^2 v}{dx^2} + F(x,\epsilon) \tfrac{dv}{dx} + G(x,\epsilon) v = 0, \quad x \in (\alpha, \beta), \\ &v(\alpha,\epsilon) = A, \quad v(\beta,\epsilon) = B, \end{aligned}$$

was considered by Sibuya [118], where $F(x,0)$ has first-order zero at x_0, $\alpha < x_0 < \beta$, and $\left[\frac{\partial f(x,0)}{\partial x}\right]_{x=x_0} < 0$. Matkowsky [86] has shown that the solution v of the above problem tends to zero in $\alpha < x < \beta$ as $\epsilon \to 0^+$, unless the coefficients of the differential equation satisfy a certain infinite sequence of equalities. In some special cases, on the other hand, it is known that if these conditions are met, the solution v has a finite nonzero limit in $\alpha < x < \beta$ as $\epsilon \to 0^+$. The author has shown that this is generally valid as long as the differential equation coefficients in a certain region of the complex x-plane are holomorphic. Error needs to be exponentially small for the problem under consideration, but the divergent series that occur in the theory of the singularly perturbed differential equations are known to generate an error smaller than the order of any power of ϵ only as $\epsilon \to 0^+$. A Phragmen-Lindelof theorem has been shown to achieve this. Given the sectors $\Im_j = \{\epsilon; a_j < \arg\epsilon < b_j,\ 0 < |\epsilon| < \rho\}$ $(1 \le j \le v)$ and function $\delta_j (1 \le j \le v)$, the author has shown that:

1. $U_j \Im_j = \{\epsilon; 0 < |\epsilon| < \rho\}$,
2. δ_j is holomorphic in $\Im_j$,
3. δ_j is asymptotically zero as $\epsilon \to 0$ in $\Im_j$,
4. $|\delta_j(\epsilon) - \delta_k(\epsilon)| \le c_0 \exp\left\{\frac{-c_1}{|\epsilon|^\lambda}\right\}$ in $\Im_j \cap \Im_k$ for some positive number c_0, c_1 and λ, whenever $\Im_j \cap \Im_k \neq \emptyset$, we have $|\delta_j(\epsilon)| \le c_2 \exp\left\{\frac{-c_1}{|\epsilon|^\lambda}\right\}$ in $\Im_j$ for some positive number c_2. This result is used to demonstrate that the Matkowsky condition [86] implies resonance in the sense of Kopell [69], if D_0 is a disk with a center at $x = 0$, i.e., $D_0 = \{x; |x| < r_0\}$ for any $r_0 > 0$.

The asymptotic solution of the linear boundary value problem of the form

$$\begin{aligned} &\epsilon^2 y'' + [xp(x) + \epsilon^2 f_1(x,\epsilon)]\, y' + g_1(x,\epsilon) y = 0, \\ &y(a) = A, \quad y(b) = B, \quad a < 0 < b, \end{aligned} \tag{1.3.4}$$

was considered by Lewis [76] where $p(x) < 0$, p, f and g are analytical. In both resonant and non-resonant cases, he studied the asymptotic nature of the solution of the comparison equation and found that small changes in one of the coefficients in the equation could lead to significant changes in the solution. It showed that if $\frac{-g_1(0,0)}{p(0)} \notin \mathbb{N}$, then resonance does not occur and $O(\epsilon^n)$ is the solution for closed subintervals of (a, b) for $n \in \mathbb{N}$ with possible boundary layer behaviour at the end points. If $\frac{-g_1(0,0)}{p(0)} \in \mathbb{N}$, then, depending on the nature of f_1, g_1 and p, resonance may or may not occur. Transformations of dependent and independent variables are performed alternately in order to find the necessary conditions for the occurrence of the resonance until (1.3.4) is reduced to an equation of the form

$$\epsilon^2 y'' + [-x + \epsilon^{2m} f(x,\epsilon)]\, y' + [\beta_m(\epsilon) + \epsilon^{2m-2} g(x,\epsilon)]\, y = 0,$$

where m is an arbitrary positive integer and

$$\begin{aligned} &\beta_m(\epsilon) = \sum_{i=0}^{2m-3} \epsilon^i c_i, \quad a = a_0 + O(\epsilon^2), \quad b = b_0 + O(\epsilon^2), \\ &A = A_0 + O(\epsilon^2), \quad B = B_0 + O(\epsilon^2). \end{aligned}$$

A sequence of necessary conditions for resonance was generated by the author and found equivalent to those proposed by Cook and Eckhaus [24]. The author provided a counterexample to the result of Olver [104] on the occurrence of resonance, that is, an example is given that is known to possess resonance but does not satisfy the conditions given in

Olver [104] for sufficiency.

The sufficiency of the Matkowsky [86] condition concerning the differential equation (1.3.2) was investigated by Lin [78] assuming that $f(0, \epsilon) = 0$ is equivalent in ϵ, $f_x(0, \epsilon) \neq 0$ with $f > 0$ for $x < 0$ and $f < 0$ for $x > 0$. The author generalized the findings of Sibuya [118] and considered the case where D is a disc containing the real interval $[-a, b]$. He has shown that transformations take (1.3.2) to

$$\epsilon y'' - 2xy' + (p - 1 + \delta_j(\epsilon))y = 0$$

in each of the subdomains

$$x \in D, \quad \epsilon \in S_j = \{\epsilon : a_j < \arg \epsilon < b_j, 0 < |\epsilon| < \rho_0\} \; (j = 1, 2, ..., v)$$

and thus

$$|\delta_j(\epsilon)| \leq H_j \exp\left(\frac{-r^2}{|\epsilon|}\right) \text{ for } \epsilon \in S_j$$

is obtained using the results of Sibuya [118]. These estimates of $\delta_j(\epsilon)$ are not good enough for resonance, since the radius r of such a disc is generally small. Using co-homological methods and the generalization of the Phragmen-Lindelof theorem, $\delta_j(\epsilon)$ estimates are enhanced.

Lange and Miura [73] considered the first-order turning point problem

$$\epsilon^2 y''(x) + q(x)y(x) + \alpha(x)y'(x - 1) + \beta(x)y(x - 1) = \psi(x),$$

subject to boundary conditions $y(x) = \phi(x)$ on $-1 \leq x \leq 0$, $y(l) = \gamma$, for q, α, β, ϕ, ψ given as continuously differentiable functions of x on $[0, l]$, γ, and $l > 1$ are constants independent of ϵ and $q(x) > 0$ to consider only rapid oscillations. The coefficient $q(x)$ has the form $q(x) = (\xi - x)p(x)$ where $p(x)$ has constant sign in $0 < x < l < 2$ and $\xi \in (0, l - 1)$. Thus, ξ is a first-order turning point. To cover such problems, methods from [72, 74] are extended by a matching across the turning point. Due to the inclusion of the delay terms, the non-uniform behaviour at $\xi = x$ as $\epsilon \to 0$ implied similar non-uniformities at $x = \xi + 1$.

Donnell [103] provided sufficient conditions for the solution to exist and examined its asymptotic behaviour as $\epsilon \to 0^+$ for the following nonlinear boundary value problem

$$\epsilon y'' = f(t, y, y'), \quad -1 < t < 1, \quad y(-1) = A, \quad y(1) = B,$$

when some or all components have a turning point at $t = 0$, where y, f, A, and B are in $\Re^n$. If $\frac{\partial f_i}{\partial y_i'}$ vanishes at $t = 0$, some or all of the components possess a turning point. The relevant reduced problem

$$f(t,y,y')=0,\quad -1<t<1,\quad y(-1)=A,\ \text{or}\ y(1)=B,$$

(or perhaps none of these) is assumed to have a smooth solution $u=u(t)$ and the reduced solution will satisfy

$$0=f_i\left(t,y_1,...,y_{i-1},u_i,y_{i+1},...,y_n,y_1',...,y_{i-1}',u_i',y_{i+1}',...,y_n'\right)$$

for t in $[-1,1]$, y_j to be defined in some region D_j, and $\left|y_j'\right|<\infty$, $j\neq i$. Based on the behaviour of $\frac{\partial f_i}{\partial y_i'}$ at the turning point $t=0$, the ith component of the solution $y_i(t,\epsilon)$ is shown to exhibit two types of behaviour. If $\frac{\partial f_i}{\partial y_i'}$ changes its sign in passing through $t=0$, $y_i(t,\epsilon)$ will behave differently on opposite sides of $t=0$ and the change will take place in the transition layer. In this case, the component $y_i(t,\epsilon)$ is uniformly approximated to order ϵ at the interval to the left of the transition layer by the ith component of the reduced solution satisfying the condition of the left hand boundary and to the right of the layer $y_i(t,\epsilon)$ behaves to order ϵ as the ith component of the reduced solution satisfying the condition of the right hand boundary. If $\frac{\partial f_i}{\partial y_i'}$ has the same sign for t in $(-1,1)$, then $y_i(t,\epsilon)$ has a boundary layer at one of the end points in the ith component, but no interior layer exists. In this case the position of the boundary layer depends on the sign of $\frac{\partial f_i}{\partial y_i'}$. If $\frac{\partial f_i}{\partial y_i'}$ is negative (with the exception of the turning point), the layer appears at the end point on the left, and at the right if positive.

By comparing uniform approximations for the pure singular point problem and the turning point problem, Wazwaz and Hanson [136] constructed asymptotic approximations. The general solution $u=u(x;s,\epsilon)$ of the second-order ordinary differential equation was analysed with the boundary conditions of the mixed type:

$$\begin{aligned}&\tfrac{1}{2}\exp(x;\epsilon)u_{xx}+x(a-x)q(x;\epsilon)u_x=su,\quad 0<x<b,\\&u(0;s,\epsilon)=u(0),\quad u_x(b;s,\epsilon)=u_x(b),\end{aligned}\tag{1.3.5}$$

where $0<a<b$, $p>0$ and $q>0$. As $\epsilon\to 0$, the functions $p(x;\epsilon)$ and $q(x;\epsilon)$ are positive, analytic and have uniform asymptotic expansions of the form

$$p(x;\epsilon)\sim\sum_{m=0}^{\infty}\epsilon^m p_m(x),\quad \textit{and}\quad q(x;\epsilon)\sim\sum_{m=0}^{\infty}\epsilon^m q_m(x).$$

Here, $x=a$ is a second-order turning point and $x=0$ is a regular singular point. For a population that increases logistically in the absence of random perturbations, but is subject to demographic type stochasticity, the study of the above type of problem was inspired by the study of the first passage time problem. In the form of uniform reduction theorem [106], a uniform approximation about the singular point is constructed in such a way that the approximation is uniformly true in the subdomain $(0, c_1)$, where c_1 is any constant in $(0, a)$. The two linear combinations of Whittaker functions is generated by this approximation with one exponentially small function $(W_1(k; z))$ and the second with only exponentially large terms $(W_2(k; z))$. When $x \in [c_2, b)$, where c_2 is another constant in $(0, a)$, the uniform approximation of the turning point obtained in terms of parabolic cylindrical functions is uniformly valid. An interval in which both expansions overlap is obtained by choosing $c_2 < c_1$. In an overlap region, strictly dominant and recessive components of both restricted uniform approximations are formally matched independently to produce a complete approximation of the general solution.

An asymptotic study of the following singularly perturbed second-order ordinary differential equation was proposed by Wazwaz [133] with two second-order interior turning points

$$\epsilon p(x)u_{xx} + 2(x-a)(b-x)q(x)u_x - 2su = 0, \quad 0 < x < 1, \quad 0 < a < b < 1,$$

for appropriate boundary conditions. The uniform reduction theorem [3] is used to construct first-order uniform asymptotic approximations in ϵ. Firstly, an approximation is obtained over the subdomain $x > a$ that covers $x = b$ but excludes $x = a$, which is linear combination of parabolic cylindrical functions U and V. The argument used for U and V, at point $x = b$, is the stretched variable Z. As it breaks down at $x = a$, the approximation fails to cover the entire domain. A first-order approximation which is valid in the subdomain $x < b$, covering $x = a$ but excluding $x = b$, is constructed essentially analogous to the derivation of the previous case. This is also a linear combination of U and V at the turning point $x = a$ with stretched variable Y. At $x = b$, the resultant approximation breaks down. Using the overlapping domain of validity between $x = a$ and $x = b$, the exponential and non-exponential components of leading and first-order terms of both approximations are matched. The author also believed that it is possible to treat the given problem as an eigenvalue problem with s being the parameter of eigenvalue. The matched approximation approach is used to study the resonance parameters associated with each turning point and the combined effect of both turning points. It turns out that the key eigenvalues are unique, although there are two turning points.

An explicit asymptotic approximation of the solution of the problem (1.3.1) was presented by Harris and Shao [50] for sufficiently small ϵ. The method of upper solutions $(\beta(x, \epsilon))$ and lower solutions $(\alpha(x, \epsilon))$ that connect the solutions is used to derive these asymptotic approximations. The approximations demonstrate that the solution is capable of showing boundary layers at one or both end points or an interior shock layer for which $\lim\limits_{\epsilon \to 0^+} \{\beta(x, \epsilon) - \alpha(x, \epsilon)\} = 0$ uniformly covers the entire domain.

Wazwaz [135] examined an asymptotic analysis of the problem

$$(b^2 - t^2)\left[\epsilon p(t)u_{tt} + (t^2 - a^2)q(t)u_t\right] = su,$$

with a pair of regular singular points $t = \pm b$ and a pair of interior second-order turning points at $t = \pm a$. Four leading-order uniform approximations were constructed by the author, each restricted to a region containing one critical point, and then neighboring approximations are formally matched independently on an overlapping domain that yields an asymptotic approximation to leading order of the general solution.

In [134], Wazwaz extended above mentioned study in the case of the arbitrary number of turning points The problem

$$\epsilon f(x)u'' + 2P(x)q(x)u' + su = 0, \quad x \in (0, 1)$$

was known to be $u(0)$, $u(1)$ prescribed. $f(x)$ and $q(x)$ are assumed to be positive and bounded away from zero, ϵ is a small positive parameter, $P(x)$ is a polynomial with an arbitrary number of real simple zeros and s is a variable quantity that is known as an eigenvalue. To construct a set of asymptotic representations that are formally matched on overlapping domains, the uniform reduction method [106] is used. In the large asymptotic argument of the parabolic cylindrical functions, the exponential and algebraic terms emerging are separately matched and the resulting asymptotic representations are merged into a single composite approximation.

Skinner [119] has considered the turning point problem

$$\epsilon^2 y'' + \left[xa(x) + \epsilon b(x) + \epsilon^2 c(x, \epsilon)\right] y = 0, \quad x \in [-1, 1],$$

where $a(x) > 0$ on $[-1, 1]$ and $a(x)$, $b(x)$, $c(x, \epsilon)$ are perfectly smooth functions. Asymptotic approximations are constructed using a formal method of matched asymptotic expansions [71] by incorporating a transformation given by

$$y(x, \epsilon) = \exp\left[\frac{i}{\epsilon}\int\limits_0^x \sqrt{|t|\, a(t)}dt\right] z(x, \epsilon).$$

A formal composite expansion is constructed by generating few terms in the formal inner and outer expansions. The problem of proving that these formal expansions for $z(x, \epsilon)$ and the corresponding asymptotic approximations for $y(x, \epsilon)$ are uniformly valid is being investigated.

Nakano [95] studied third-order linear singularly perturbed ordinary differential equation

$$\epsilon^3 y''' = \sum_{k=1}^{3} \epsilon^{3-k} p_k \left(x - \frac{\epsilon}{x^2}\right)^k y^{3-k}; \quad (0 < |x| \leq x_0,\, 0 < \epsilon \leq \epsilon_0)\,, \tag{1.3.6}$$

where x_0 and ϵ_0 are positive constants, x is a complex variable and $p_1 = a_1 + a_2 + a_3$, $p_2 = -(a_1a_2 + a_2a_3 + a_3a_1)$ and $p_3 = a_1a_2a_3 \forall a_k \neq 0$, $a_1 < a_2 < a_3$. Here, $x = 0$ is called a turning singular point because it is turning as well as singular point of the problem (1.3.6). Differential equations with turning point are defined by their characteristic polygon (introduced by Sibuya [58]), but in the case of turning singular points, this approach is unsuccessful, so the present work adopts a modified method. The domain is subdivided into the inner and outer region, reduced equation is obtained, and internal and external WKB solutions are developed. The relationship between the external and internal solution, represented by a matrix called the matching matrix, is determined by taking appropriate points belonging to both (inner and outer) regions. The author demonstrates that the maximum existence regions of the inner and outer solutions are bounded by stokes curves. The boundaries of the existence regions of the outer solution are determined by those of the maximum existence region of the inner solutions, which are called canonical regions. Nakano [96] has used the same method to test the nth order linear differential equation

$$\epsilon^{nh} y(n) = \sum_{k=1}^{3} \epsilon^{(n-k)h} p_k(x,\epsilon) y^{n-k}; \quad (0 < |x| \leq x_0,\, 0 < \epsilon \leq \epsilon_0)\,, \tag{1.3.7}$$

that satisfies the same conditions as in [95] and coefficients $p_k(x,\epsilon) = p_k\left(x^m - \frac{\epsilon^l}{x^r}\right)^k$, $k = 1, 2, ..., n$ where m, l and r are positive integers such that $h > \frac{(m+1)l}{(m+r)}$ and p_k satisfy conditions that ensure that the characteristic equation of (1.3.7) has characteristic roots which coincide with $x = 0$.

The differential equation

$$\begin{aligned} &\epsilon y'' + a(x)y' + b(x)y = 0, \quad y(x_+) = A, \quad y(x_-) = B, \\ &x \in [x_-, x_+], \quad x_- < 0 < x_+, \end{aligned}$$

was considered by Wong and Yang [138] where $a(x)$ and $b(x)$ are sufficiently smooth near $x = 0$ and behave asymptotically as αx and β respectively, for some real numbers, α and β. The authors addressed the case where $\frac{\beta}{\alpha}$ is not a positive integer and $\alpha > 0$. The method proposed in this paper is similar to that given by Bender and Orszag [13], who in the typical case applied the method of matched asymptotic expansion when $x_- = -1$ and $x_+ = 1$ to construct a fairly straightforward explicit asymptotic solution involving parabolic cylinder functions. However, by transforming the given differential equation to the Liouville form, converting it to an integral equation and using the method of successive approximations, more rigorous evidence is given. The authors viewed the case of $\alpha < 0$ and $\frac{\beta}{\alpha}$ a non-negative integer in the companion paper [137]. The original differential equation in y and x is translated into variables U and t as

$$M[U] = l_1(\epsilon, t)y''U + l_2(\epsilon, t)U,$$

where M is a second-order differential operator whose coefficients are formulated from those of the original equation. The equation $M[U] = 0$ is solved in terms of parabolic cylinder functions and the asymptotic behaviour of the functions $l_1(\epsilon, t)$ and $l_2(\epsilon, t)$ as $\epsilon \to 0$ tending to zero is obtained. By means of the method of parameter variance, the non-homogeneous equation is converted into an integral equation, which is further solved by method of successive approximation. The final outcome gives a uniform asymptotic approximation to the unique solution $y(x)$ of the original problem as $\epsilon \to 0$ on $[x_-, x_+]$.

Ackerberg-O'Malley resonance [3] was revisited by Wong and Yang [139] by obtaining simple uniformly valid asymptotic expansions for the solution y when:

1. $\alpha > 0$, $\frac{\beta}{\alpha} = 1, 2, ...$, or
2. $\alpha < 0$, $\frac{\beta}{\alpha} = 0, -1, -2, ...$

They showed that in both the cases, the solution will produce resonance as the solution can expand exponentially in a subinterval of $[x_-, x_+]$ for the case (1), while in a subinterval of $[x_-, x_+]$ it is not exponentially small for the case (2).

In singularly perturbed advection-diffusion-reaction equations, Knaub and O'Malley [68] examined the motion of internal layers. For a small positive parameter ϵ, which tends to zero on a finite spatial domain, the authors obtained the asymptotic solution of the following PDE with a single extremely slowly moving internal layer

$$u_t = \epsilon^2 u_{xx} + \epsilon g(u)u_x + h(u),$$

where g, h are smooth, u_L, $u_R(u_L < u_R)$ satisfies $h(u_L) = h(u_R) = 0$. Apart from this smooth and compatible initial data, $u_L \leq u(x, 0) \leq u_R$ for $0 < x < 1$ constant Dirichlet data $u(0, t) = u_L$, $u(1, t) = u_R$ for $t > 0$ is prescribed. A monotonically increasing shock profile function $u_p(z)$ depending upon the $O(\epsilon)$-stretched coordinate $z = \frac{1}{\epsilon}(x - x_\epsilon(\sigma))$, centered at a slowly moving shock location $x_\epsilon(\sigma)$, is required to approximate the limiting solution after a suitable interval of time. For advection-diffusion and reaction-diffusion equations, the condition for the existence of the profile function $u_p(z)$ is given. The motion equation of the internal layer position is also approximately derived and the two cases, namely exponential asymptotics and algebraic asymptotics, are analysed.

Yang [140] investigated a singular perturbation problem with two second-order turning points given by

$$\begin{aligned}&\epsilon p(x)u_{xx} + 2(x-a)(b-x)q(x)u_x - 2su = 0,\\&0 < x < 1, \quad 0 < a < b < 1,\end{aligned} \tag{1.3.8}$$

where $p(x)$ and $q(x)$ are sufficiently smooth and strictly positive functions. Two second-order turning points are at $x = a$ and $x = b$ on the interval $(0,1)$. For the construction of two linearly independent solutions for the Eq. (1.3.8) on each of the subintervals $(0,b)$ and $(a,1)$, the author used his previous results regarding the problem with one turning point [138, 139]. A matching technique is employed to obtain a uniform asymptotic solution for Eq. (1.3.8) over the entire interval $(0,1)$. The matching occurs in the overlap interval (a,b). Only the dominant terms are kept in the asymptotic approximations of the solution of the boundary value problem for determining the value of the constants involved and boundary conditions are applied to it. The author derived a uniform asymptotic expansion for the solution over many different parts of the interval $[0,1]$, in order to clearly derive the asymptotic behaviour. As $\epsilon \to 0$, the solution approximates to the solution of the reduced problem

$$(x-a)(b-x)q(x)u_x - 2su = 0, \quad u(0) = k_1, \quad x \in [0,a).$$

Then, near the turning point $x = a$, an interior layer of thickness $O(\sqrt{\epsilon})$ occurs. The solution is exponentially small in $(a,1)$ and near the endpoint $x = 1$ a boundary layer of thickness $O(\sqrt{\epsilon})$ appears.

1.3.2 Numerical Approach

This section discusses the numerical techniques developed over a period of time to find an approximate solution of the singularly perturbed turning point problems. In addition, it is presumed that h denotes the step-size in the spatial direction, unless or otherwise specified.

Hemker [52] presented numerical treatment of the two-point boundary value problem

$$\epsilon y''(x) + f(x)y'(x) - g(x)y = k(x), \quad y(a) = \alpha, \quad y(b) = \beta,$$

where the function $g(x) \geq 0$. The first derivative $y'(x)$ is approximated by a weighted average of the forward and backward divided differences in the finite-difference discretisation and $y''(x)$ is approximated by the standard three-point difference operator. By choosing the mesh spacing and the weight factors properly at each mesh point, the author acquired a technique that combined the advantage of known methods with a low convergence order.

Miranker and Morreeuw [92] performed a semi-analytic numerical study of the stiff boundary value problem

$$\epsilon y'' + f(x)y' - g(x)y = h(x), \quad 0 < x < 1, \quad y(a) = \alpha, \quad y(b) = \beta,$$

where $f(x)$ and $g(x)$ are smooth functions. The number of turning points x_i is assumed to be finite, i.e. $f(x_i) = 0$ but $f'(x_i) \neq 0$ and $\frac{g(x_i)}{f(x_i)}$ is non-integer. A combination of differential approximation and analytical continuation theory that relates the solutions on both sides of the turning point solves the given problem.

Barrett [10] expanded the approach of upwind differences to cover internal turning point problems and outlined some of the reasons behind the numerical difficulties that Dorr and Parter [27] and others had previously experienced. Proposed approximations of finite-difference are considered accurate to exponentially small terms. The difference between homogeneous and non-homogeneous problems is also shown to be important.

For the following turning point problem,

$$\epsilon u''(x) + axu'(x) - bu(x) = f(x), \quad x \in (-1, 1), \quad u(-1) = \alpha, \quad u(1) = \beta,$$

where a, b, $\epsilon \in \Re^-$, $\lambda = \frac{b}{a} \notin Z$ and $f(x)$ is sufficiently smooth in $C^5(-1, 1)$, Farrell [32] proved ϵ-uniform convergence of number of finite-difference schemes. The internal layer of the cusp form exhibits these types of problems at $x = 0$. The difference scheme is described by

$$L^h u_i \equiv \epsilon \sigma_i D_+ D_- u_i + x_i a D_0 u_i - b u_i = f_i, \quad i = 1, 2, ..., n-1,$$

where n is even, $f_i = f(x_i)$ and $\sigma_i = \sigma_i \left(\frac{a x_i h}{2\epsilon}\right)$ is a positive parameter under consideration to be defined based on the form of finite-difference scheme. $\sigma_i \geq \frac{a|x_i|h}{2\epsilon}$ and $|\sigma_i - 1| \leq C\frac{a|x_i|h}{2\epsilon}$ are assumed such that the scheme becomes of a positive type and satisfies maximum principle. When applied to the turning point problem, the number of finite-difference schemes, including upwinding, exponential fitting and Hemker's scheme, was found to be uniformly convergent. He discovered that $|u(x_i) - u_i| \leq Ch^{\min(\lambda, \frac{4}{5})}$ is for upwinding and Hemker's scheme while $|u(x_i) - u_i| \leq Ch^{\min(\lambda, 1)}$ is for exponential fitting.

In [33, 34], Farrell suggested suitable conditions for uniform convergence of various difference schemes for singularly perturbed turning and non-turning point problems.

Kellog [65] applied Allen-Southwell [5] exponentially fitted finite-difference numerical scheme to find numerical solution of the singular perturbation problem

$$-\epsilon u'' + pu' + qu = f, \quad -1 < x < 1,$$

with $u(\pm 1)$ prescribed, $q > 0$ and p vanishing at a finite number of points in the domain called stagnation points or turning points. For the proposed scheme, such hypotheses are made and error estimates are obtained, which are known to be uniformly convergent.

Liseikin [83] has developed a special difference net on non-uniform grids that are compressed in the boundary layer region of the turning point problems

$$Lu \equiv \epsilon^2 u'' + xa(x)u' + c(x)u = f(x), \quad -1 < x < 1, \quad u(0) = u(1) = 0,$$

and $Lu = f, \quad -1 < x < 1, \quad u(-1) = u(1) = 0$, with $a(x) \geq a_0 > 0$ and $c(x) \geq c_0 > 0$. With no domain constraint of $f(0)$ and $\frac{c(0)}{a(0)}$, it has been shown that the simplest monotone scheme on the resulting net converges uniformly for first-order accuracy with respect to the perturbation parameter. In addition, Liseikin [84] studied external and internal boundary layers of second-order two-point boundary value problems and demonstrated results on a transformation leading to uniformly convergent solutions.

Berger et al. [14] derived a priori estimates and presented a numerical solution with turning points and Dirichlet type boundary conditions for the following two-point boundary value problem

$$\begin{aligned} &Ly \equiv -\epsilon y''(x) + p(x)y'(x) - q(x)y(x) = f(x), \quad -1 < x < 1, \\ &y(-1) = d_1, \quad y(-1) = d_1, \quad y(1) = d_2. \end{aligned} \tag{1.3.9}$$

Here, d_1 and d_2 are given constants, q and f are required to be in $C^1[-1,1]$, p is assumed to be in $C^2[-1,1]$, p is allowed to have a finite number of zeros located at the points $z_1, ..., z_r$ in $(-1,1)$ and $q(x)$ is assumed to be positive to exclude the so-called resonance case. As $\epsilon \to 0^+$, the authors have shown that the behaviour of the solution $y(x)$ near a turning point z_i depends upon the sign of the constant $\beta_i = \frac{q(x_i)}{p'(z_i)}$. They showed that the study of the problem with an arbitrary number of turning points can be reduced to one turning point through some preliminary observations concerning (1.3.9) located at $x = 0$. They proved that the solution $y(x)$ is smooth near $x = 0$ for $\beta < 0$, on the other hand if there is an interior layer at $x = 0$ for $\beta > 0$, the nature of which depends on the nature of β. For $y(x)$ and its derivatives, priori bounds are proven. In general, y is seen to have a boundary layer at $x = -1(1)$ if and only if $p(-1) > 0(p(1) < 0)$. The authors used Mistikawy and Werle [31] scheme for the numerical approximation of the solution with some modifications. They proved that for the proposed numerical scheme; (i) $O(h)$ accuracy is obtained uniformly in ϵ for $\beta < 0$ (ii) $O(h)$ outside the turning point region for $\beta > 0$ while in the turning point region it is $O\left(h \ln \frac{6}{ch^2}\right)$ for $\beta = 1$ and $O(h^\beta + h)$ whenever $\beta > 0$, $\beta \neq 1$.

Farrell [35] gave adequate conditions for uniform convergence of a difference scheme for singularly perturbed turning point problem that are met by a broad class of schemes and developed a class of difference schemes for the problem presented. The problem (1.3.9) with single turning point at the origin was considered by the author. The characteristic parameter β is treated as a non-integer. If $u(x)$ is the solution of the Eq. (1.3.9) and u_i^h is the solution of

$$L_\epsilon^h u_i^h \equiv \epsilon_i^{\pm} D_+ D_- u_i^h + a_i^h D_\pm u_i^h - b_i^h u_i^h = f_i^h, \quad i \in (N, N), \quad u_{-N}^h = 0 = u_N^h,$$

it is proved that the coefficients $\epsilon_i^{\pm}$, $a_i^h = \alpha_i a(x_i)$, $b_i^h = \beta_i(x_i)$ and f_i^h are bounded such that the scheme is uniformly stable, i.e.,

$$\begin{aligned} &\epsilon_i^{\pm} > 0, \quad \alpha_i \geq \underline{\alpha} > 0, \quad \beta_i \geq \underline{\beta} > 0, \quad \left|a_i^h - a(x_i)\right| \leq Ch, \\ &\left|b_i^h - b(x_i)\right| \leq Ch, \quad \left|f_i^h - f(x_i)\right| \leq Ch, \quad \left|\epsilon_i^{\pm} - \epsilon\right| \leq Ch\ (|a(x_i)| + h)\,; \end{aligned}$$

the error estimate is given by $\epsilon \leq \epsilon_0$, $\left|u_\epsilon(x_i) - u_i^h\right| \leq Ch^{\min(\beta,1)}$, $-1 \leq x_i \leq 1$, for $h \leq h_0$. The author showed that some well known schemes, e.g., Il'in-Allen-Southwell scheme [5, 57], Abrahamsson scheme [1], complete exponential fitting schemes, simple upwind scheme, Samarskii scheme, fulfill the above-mentioned sufficiency condition, while these estimates are not satisfied by centered differences and scheme proposed by Abrahamsson [1] for nonlinear problems. It is found that the rate $\min(\beta, 1)$ is the best attainable convergence rate for a general scheme of this type (that is, those satisfying sufficiency condition). Furthermore, Farrell gave a uniform convergence result for a turning point problem in [36].

For singularly perturbed convection-diffusion equation with turning points, Hedstrom and Howes [51] presented domain decomposition methods. The problem

$$-xu' = \epsilon \Delta u(x) \text{ on } \Omega,$$

which is a square in the plane, was considered. Asymptotic analysis is used to show that the solution of the problem almost satisfies the reduced equation $u_x = 0$ in subdomains that are at least a distance of $C\sqrt{\epsilon}$ from $\partial\Omega$, while boundary layers which exist in the vicinity of $\partial\Omega$ depending on the boundary values. In order to determine the partition into subdomains and to suggest the basic functions to be used in a finite-element formulation of this problem, the authors used asymptotic information.

Vulanovic [126] used the continuous solution properties to generate numerical methods for the problem of singularly disturbed mildly nonlinear turning points. On non-uniform meshes, he used a finite-difference scheme and obtained comparison results using various methods.

Vulanovic developed a L^1-stable finite-difference scheme for linear second-order singularly perturbed turning point problem in [127] by using continuous change in upwinding technique based on cell Reynolds number.

The results obtained in [126] was extended by Vulanovic and Lin [129] to solve a quasilinear singularly perturbed boundary value problem with a turning point of the attractive type for developing a robust numerical scheme.

Lin [79] provided numerical treatment of quasilinear singularly perturbed boundary value problem with turning point and Dirichlet data

$$Ly = \epsilon y'' + b(x,y)y' - c(x,y) = 0, \quad -1 < x < 1, \quad y(-1) = A, \quad y(1) = B, \tag{1.3.10}$$

where $b(x,y), c(x,y) \in C^2([-1,1] \times \Re)$, $b(0,y) = 0$, $b_x(0,y) < 0$ for $|y| \leq r+1$, $b(x,y) \neq 0$ for $|y| \leq r+1$ and $x \neq 0$, and $c(x,y) \geq c_0 > 0$ on $[-1,1] \times \Re$. Here, c_0 is a positive constant and r is a finite constant to be determined. A numerical method proposed by the author is convergent in the maximum norm for arbitrary $\epsilon > 0$. First, bounds on the solution and its derivatives are obtained for the problem (1.3.10) for the construction of the numerical scheme and then an initial value problem is developed that approximates the problem (1.3.10). Afterwards, to solve the initial problem, a finite-difference scheme on non-uniform mesh is constructed and the error estimate $\max_{-N \leq i \leq N} \left| y(x_i) - u_i^h \right| \leq Ch^{s/(1+r)}$ is obtained. An improvement is made in the error estimation by indicating that if there exist a method solving (1.3.10) and the order of convergence $O(h^s/\epsilon^r)$, $s > 0$, $r > 0$, then this method can be used for $\epsilon > h^{s/(1+r)}$, while the above algorithm can be used for $\epsilon \leq h^{s/(1+r)}$, and the error estimate in that case will be $O\left(h^{s/(1+r)}\right)$.

In [80], a uniformly convergent difference scheme was constructed for a semilinear turning point problem. The author considered the following boundary value problem

$$\epsilon u'' + p(x)u' - f(x,u) = 0, \quad -1 < x < 1, \quad u(-1) = A, \quad u(1) = B,$$

where in the domains $[-1,1]$ and $[-1,1] \times \Re$, the functions $p(x)$ and $f(x,u)$ are sufficiently smooth under the assumption $f_u(x,u) \geq f^* > 0$, $(x,u) \in [-1,1] \times \Re$; $p(0) = 0$, $p'(0) < 0$ and $p(x) \neq 0$ when $x \neq 0$. For the nonlinear differential equation, taking these assumptions into account, an iterative process converging independently of ϵ is constructed.

For the following linear second-order singularly perturbed BVPs with turning points, Lopez [85] discussed the stability of a three point scheme

$$\epsilon y''(t) + s(t)y'(t) + c(t)y(t) = f(t), \quad t \in [a,b], \quad y(a) = \eta, \quad y(b) = \zeta,$$

with $s(t)$, $c(t)$, $f(t)$ be continuous functions on the integration interval $[a,b]$; $c(t) \leq 0$ and

$s(t)$ may change sign. The author considered the form

$$y_{n+1} - \gamma_n y_n + \beta_n y_{n-1} = b_n, \quad n = 1, ..., N,$$

to be a three-point scheme (with variable step size) where $y_0 = \eta$, $y_{N+1} = \zeta$, $t_0 = a$, $t_{n+1} = b$, $y_n \cong y(t_n)$ with $t_n = t_{n-1} + h_{n-1}$, h_{n-1} for $n = 1, ..., N+1$. The values β_n, b_n, γ_n for $n = 1, ..., N$ are given by

$$\beta_n = \frac{h_n}{h_{n-1}} \frac{2\epsilon - s_n h_n}{2\epsilon + s_n h_{n-1}}, \quad b_n = \frac{h_n \left(h_n + h_{n-1}\right)}{\left(2\epsilon + s_n h_{n-1}\right)} f_n,$$

$$\gamma_n = \frac{\left(h_n + h_{n-1}\right) \left[2\epsilon + s_n \left(h_{n-1} - h_n\right) - c_n h_n h_{n-1}\right]}{h_{n-1} \left(2\epsilon + s_n h_{n-1}\right)}.$$

The stability analysis of the above scheme is carried out by studying the stability of a LU factorization for the coefficient matrix of the above tridiagonal system, which depends on the diagonal term α_n of the matrix L since it is assumed that diagonal terms of U is 1. In addition, the study of behaviour of the diagonal terms of L is equivalent to the study of the solution of the discrete Riccati equation $\beta_n x_{n+1} x_n - \gamma_n x_{n+1} + 1 = 0$ for $n = 1, ..., N$ where $x_0 = 0$. A good mesh selection strategy for the treatment of boundary and interior layers was also proposed by the author, which provides a mesh on which the three-point system is stable.

Vulanovic and Farrell [128] presented numerical analysis of the following attractive multiple boundary turning point problem

$$Lu := -\epsilon u'' - x^k b(x) u' + c(x) u = f(x), \quad x \in I = [0,1], \quad (u(0), u(1)) = [U_0, U_1], \tag{1.3.11}$$

where U_0 and U_1 are given numbers. They further presumed that $k = 2$ or $k \in [3, \infty)$; $b, c, f \in C^3(I)$,$b(x) \geq b_0 > 0$,$x \in I$; $c(x) \geq 0$,$x \in I$; $c(0) > 0$, $x \in I$; $a(x) := x^k b(x)$. Estimates on the derivatives of the solution u_ϵ are given after proving existence and uniqueness of the solution, which are further defined following a more precise representation of the solution

$$u_\epsilon(x) = w v_\epsilon(x) + z_\epsilon(x), \quad x \in I, \quad v_\epsilon(x) = \exp(-\mu x), \quad \mu = \sqrt{\tfrac{c(0)}{\epsilon}}, \quad |w| \leq M,$$
$$\left| z_\epsilon^{(i)}(x) \right| \leq M \left(1 + \epsilon^{(1-i)/2} \exp\left(\tfrac{-mx}{\sqrt{\epsilon}}\right)\right), \quad i = 0, 1, 2, 3, \quad x \in I,$$

which are used to derive the error estimates. A first-order exponentially fitted finite-difference scheme is constructed using special non-equidistant mesh that is dense near the origin. The mesh points are given by

$$\lambda(t) = \begin{cases} \psi(t) := \sqrt{\epsilon}\frac{t}{\gamma - t}, & t \in [0, \alpha], \\ \Pi(t) := \beta(t-\alpha)^3 + \frac{\psi''(a)}{2}(t-\alpha)^2 + \psi'(\alpha)(t-\alpha) + \psi(a), & t \in [\alpha, 1], \end{cases}$$

$x_i = \lambda(t_i)$, $t_i = \frac{i}{n}$, $i = 0(1)n$ (nodal points), where α is an arbitrary number from $(0,1)$ and β is determined by $\Pi(1) = 1$, $\lambda = \alpha + \sqrt[6]{\epsilon}$, and ψ is a modification of the inverse boundary layer function v_ϵ. ψ is expanded by a polynomial at the rest of the interval, so that $\lambda \in C^2(I)$ is monotonous. The discretisation of the problem (1.3.11) for the modified scheme is obtained at mid points $x_{i+1/2} = x_i + \frac{h_{i+1}}{2}$, $1(1)n-1$. The error estimate for the improved scheme is $O(\sqrt{\epsilon}n^{-1} + n^{-2})$ with n mesh points, and is accurate for $\sqrt{\epsilon} \leq n^{-1}$ in the second-order. However, prior information about the behaviour of the continuous solution was required by this method.

Clavero and Lisbona [23] studied the following singularly perturbed nonlinear differential equations with attractive turning point

$$\begin{aligned} -\epsilon u''(x) + (xu(x))' + a(x)u(x) = f(x), \quad -1 < x < 1, \quad u(-1) = A, \quad u(1) = B, \\ -\epsilon y''(x) + (xy(x))' + b(x, y(x)) = 0, \quad -1 < x < 1, \quad y(-1) = A, \quad y(1) = B, \end{aligned}$$

where $a(x) \geq 1$, $a(0) \geq 1 + \delta$, $b_y(x, y) \geq 1 + \delta$ and $\delta > 0$. Using the results of Berger et al. [14], though they did not use parabolic cylindrical functions, bounds on the solution and its derivatives are obtained. To study the case of the semilinear problem, techniques used for linear problems are generalized. For a family of finite-difference schemes on " locally almost regular " meshes, uniform convergence is proven. Uniform convergence of the simple upwind system (in L^1-norm) is proved and then this property is transferred to a whole family of schemes similar to upwind schemes, that is Samarskii and exponential fitted methods, by arguments of continuity. The authors gave numerical results using exponentially fitted schemes for the integration of linear problem and the problem of Samarskii's semilinear scheme, and using very few points in the integration mesh, good numerical approximations are obtained in the turning point region. A uniform initial mesh is originally used and then modified by means of an iterative equi-distribution algorithm of arc length using piecewise linear interpolation of the previous discrete solution.

For second-order ordinary differential equations exhibiting twin boundary layers, Natesan and Ramanujam [99] presented a numerical method for the following singularly perturbed turning point problem

$$\epsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in D = (-1, 1), \quad u(-1) = A, \quad u(1) = B, \tag{1.3.12}$$

where a, b and f are sufficiently smooth functions. Further to

$$a(0) = 0, \quad a'(0) < 0, \quad b(x) \geq b(0) > 0, \tag{1.3.13}$$

to eliminate the case of resonance. It is assumed that

$$|a'(x)| \geq |a'(0)/2|, \quad x \in \overline{D} := [-1, 1]. \tag{1.3.14}$$

is used to make sure that there are no other turning points in the interval $[-1, 1]$. TPP (1.3.12) has a unique solution with two boundary layers at $x = -1$ and $x = 1$ under the above conditions. The interval D is divided into four sub-intervals to solve the above problem, namely, $D_1 = [-1, -1 + K_\epsilon]$, $D_2 = [-1 + K_\epsilon, -\delta]$, $D_3 = [\delta, 1 - K_\epsilon]$, $D_4 = [1 - K_\epsilon, 1]$, where $\delta > 0$ is a small number, K such that $K_\epsilon \ll 1$ and K_ϵ is the width or thickness of the boundary layers which are close to $x = -1$ and $x = 1$. Therefore, two types of problems are obtained, namely inner and outer region problems involving a four differential equation scheme. Inner region problems are solved by using exponentially fitted difference schemes (EFD) [26], whereas classical upwind difference schemes solve outer region problems. By considering the asymptotic expansion solution $u(x)$ of Eq. (1.3.12) as given in [2], boundary values for inner and outer region problems are obtained. Finally, outer and inner solutions are combined over the interval $[-1, 1]$ to obtain the approximate solution to the original problem. This is an iterative method based on the terminal points $-1 + K_\epsilon$ and $1 - K_\epsilon$.

The problem (1.3.12)-(1.3.14) was solved by Natesan and Ramanujam [98] using initial value technique (IVT) originally developed in [41] to solve the singularly perturbed non-turning point problem. For this reason, the domain of definition of ODE, that is $[-1, 1]$, is divided into three disjoint sub-intervals: $D_1 = \left[-1, \frac{-1}{2}\right]$, $D_2 = \left(\frac{-1}{2}, \frac{1}{2}\right)$, and $D_3 = \left[\frac{1}{2}, 1\right]$. First of all, a reduced problem is solved, the solution of which u_0 is used as an approximation to the solution of the problem on $\left(\frac{-1}{2}, \frac{1}{2}\right)$. The value of the solution of the reduced problem at $x = \frac{-1}{2}, \frac{1}{2}$ is taken as boundary condition for ODE (1.3.12) on D_1 and D_3. The given differential equation is then solved separately by IVT on D_1 and D_3. The approximate solution is obtained by combining the solutions of the reduced problem, the initial value problem and the terminal value problem. If v_{01i} is the numerical solution of (1.3.12) on $\left[-1, \frac{-1}{2}\right]$ and v_{02i} on $\left[\frac{1}{2}, 1\right]$ by applying the EFD scheme, then $u_0(x_i) + (A - u_0(-1))v_{01i}$, $u_0(x_i)$, $u_0(x_i) + (B - u_0(1))v_{02i}$ are approximations to $u(x)$, then the solution of TPP at the respective intervals $\left[-1, \frac{-1}{2}\right]$, $\left(\frac{-1}{2}, \frac{1}{2}\right)$, and $\left[\frac{1}{2}, 1\right]$ yields the following error estimates

$$\begin{aligned}
&|u(x_i) - [u_0(x_i) + (A - u_0(-1))v_{01i}]| \leq C(h + \epsilon), \quad x_i \in \left[-1, \tfrac{-1}{2}\right], \\
&|u(x_i) - u_0(x_i)| \leq C\epsilon, \quad x_i \in \left(\tfrac{-1}{2}, \tfrac{1}{2}\right), \\
&|u(x_i) - [u_0(x_i) + (B - u_0(1))v_{02i}]| \leq C(h + \epsilon), \quad x_i \in \left[\tfrac{1}{2}, 1\right].
\end{aligned}$$

Also, the implementation of the method for parallel architecture was discussed by the author.

For the numerical solution of singularly perturbed semilinear convection-diffusion problems with attractive turning points, Linss and Vulanovic [82] developed two finite-difference schemes (one was first-order accurate and the other was second-order accurate). For a finite-difference scheme, the authors used the generalization of the hybrid stability inequality proposed by Andreev and Savin [7] to discretise a linear singularly perturbed boundary value problem with a small positive perturbation parameter ϵ. It is shown that the maximum nodal error is bounded by a special weighted l_1-type norm of the truncation error for both the schemes and ϵ-uniform piecewise convergence is set on the Shishkin mesh using this. They considered the same problem in [81] and firstly split the solution into a regular solution component and a boundary layer component with sharp estimates for their derivatives up to the third-order and then used this decomposition to evaluate the convergence on Shishkin mesh of the proposed upwind finite-difference scheme. For the proposed scheme using hybrid stability inequality, first-order convergence in the discrete maximum norm is proven.

For the parabolic problem

$$Lu(x,t) = f(x,t), \quad (x,t) \in G, \quad u(x,t) = \varphi(x,t), \quad (x,t) \in S = S_0 \cup S^L,$$

Shishkin [116] proposed difference schemes where

$$L \equiv \epsilon^{1+\alpha} L^2 + L^1, \quad L^1 \equiv b(x,t)\frac{\partial}{\partial x} - c(x,t) - p(x,t)\frac{\partial}{\partial t}, \text{ and } L^2 \equiv a(x,t)\frac{\partial^2}{\partial x^2}.$$

The coefficient $b(x,t)$ occurring in the operator L^1 satisfies either the condition

$$b(x,t) = -x^\alpha b^1(x,t), \quad (x,t) \in \overline{G},$$

or the condition

$$b(x,t) = x^\alpha b^1(x,t), \quad (x,t) \in \overline{G}; \quad b^1_0 \le b^1(x,t) \le b^{10}, \quad b^1_0 > 0, \tag{1.3.15}$$

and the functions $a(x,t)$, $b^1(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$ and $\varphi(x,t)$ are sufficiently smooth on $\overline{G}$ and the sides of G, respectively. The $\varphi(x,t)$ is a continuous function on S, $0 < a_0 \le a(x.t) \le a^0$, $p_0 \le p(x.t) \le p^0$, $c(x,t) \ge 0$ and the parameter ϵ is set to arbitrary values in the half-open interval $(0,1]$; $S_0 = \{(x,t) : x \in \overline{D}, t = 0\}$, $S^L = S^L_1 \cup S^L_2$ where S^L_1 and S^L_2 are the left and right parts of the lateral surface S^L, α is a non-negative number and the behaviour of the solution in the neighborhood of the boundary $x = 0$ is determined by the sign of the function $b^1(x,t)$ and the value of the parameter α. The boundary layer is parabolic if $\alpha \geq 1$, if $\alpha = 0$ has a power decay, and if $\alpha > 1$ has an exponential decay. The boundary layer does not appear if $\alpha < 1$. In the neighborhood of S_2^L, the boundary layer is regular and exponential, but in the case of (1.3.15), the boundary layer only occurs in the neighborhood of S_1^L. With the use of the method of condensing grids, that is classical grid approximations on grids condensing in a neighborhood of the boundary layers and in a neighborhood of the normal part of the solution, the author developed ϵ-uniformly convergent finite-difference schemes. It is used to build schemes for $b^1 > 0$, $\alpha < 1$, piecewise uniform grids with respect to x with some intervals of constant increase in the neighborhood of the singularity of the regular part of the solution (and special grids with a slowly changing increment for the construction of a quasi-optimal scheme with respect to x). For all the schemes, several finite-difference schemes are developed and error estimates are presented.

Natesan et al. [97] applied the classical finite-difference scheme on an appropriate piecewise uniform Shishkin mesh to solve the singularly perturbed turning point problem (1.3.12)-(1.3.14) showing exponential boundary layers. The resulting fitted finite-difference scheme is given by

$$\begin{aligned} &L^N U(x_i) \equiv \epsilon \delta^2 U(x_i) + a(x_i) D^* U(x_i) - b(x_i) U(x_i) = f(x_i), \\ &x_i \in D_\epsilon^N = \{x_i : 1 \leq i \leq N-1\}, \quad U(0) = A, \quad u(1) = B, \end{aligned}$$

where $\delta^2 Z_i = \frac{2(D^+Z_i - D^-Z_i)}{x_{i+1} - x_{i-1}}$, $D^* = \begin{cases} D^+ Z_i & \text{if } a(x_i) > 0, \\ D^- Z_i & \text{if } a(x_i) < 0, \end{cases}$

$$x_i = \begin{cases} -1 + i\frac{4\tau}{N} & \text{for } 0 \leq i \leq \frac{N}{4} \\ \tau - 1 + \left(i - \frac{N}{4}\right)\frac{4(1-\tau)}{N} & \text{for } \frac{N}{4} + 1 \leq i \leq \frac{3N}{4} \\ 1 - \tau + \left(i - \frac{3N}{4}\right)\frac{4\tau}{N} & \text{for } \frac{3N}{4} \leq i \leq N \end{cases}$$

and the transition parameter is $\tau = \min\{\frac{1}{4}, k\epsilon \ln N\}$ with $K \geq \frac{1}{\min\{a_0, b_0\}}$.The benefit of this approach over the previous ones [98, 99] was that it was simple to implement the method presented in this work that did not require any knowledge about the asymptotic approximation.

Dunne et al. [28] investigated a class of singularly perturbed time-dependent convection-diffusion problems with a boundary turning point on a rectangular domain

$$
\begin{aligned}
&L_\epsilon u(x,t) \equiv (\epsilon u_{xx} + a u_x - b u_t - du)(x,t) = f(x,t) \text{ in } D = \Omega \times (0,t), \quad \Omega = (0,1)\\
&u(x,t) = g(x,t) \text{ on } \Gamma,\\
&a(x,t) = a_0(x,t)x^p, \quad p \geq 1 \;\; \forall (x,t) \in \bar{D}, \quad a_0(x,t) \geq \alpha > 0,\\
&b(x,t) \geq \beta > 0, \quad d(x,t) \geq \delta \geq 0 \;\; \forall (x,t) \in \bar{D},\\
&\Gamma = \bar{D} = \Gamma_l \cup \Gamma_b \cup \Gamma_r, \quad \Gamma_l = \{(0,t) | 0 \leq t \leq T\},\\
&\Gamma_b = \{(x,0) | 0 \leq x \leq 1\}, \quad \Gamma_r = \{(1,t) | 0 \leq t \leq T\}.
\end{aligned}
$$

Here a_0, b, d, f and g are sufficiently regular and f, g at the corners of the domain satisfy sufficient compatibility conditions. A parabolic boundary layer of width $\sqrt{\epsilon}$ for a small ϵ is used to solve this problem. It is defined that the corresponding reduced problem is

$$(a(v_0)_x - b(v_0)_t - dv_0)(x,t) = f(x,t) \text{ in } D, \quad v_0(x.t) = g(x,t) \text{ on } \Gamma_b \cup \Gamma_r. \quad (1.3.16)$$

It is assumed here that the characteristic of the reduced problem (1.3.16) does not converge with the boundary Γ_l, but gradually deviates from the lateral boundary away from the vertical. Using the standard first-order upwind finite-difference operator that satisfies the maximum principle, the problem is discretised on a layer-adapted piecewise-uniform Shishkin mesh. The time derivative using the backward finite-difference is discretised. The problem solution is subdivided into regular and singular components for the convergence analysis of the method and a comparison principle is used in combination with the appropriate barrier function. An error estimate of $O(N_x^{-1} \ln^2 N_x + N_x^{-1})$ is obtained using this decomposition and the estimates of the singular component.

Ramos [112] proposed an exponentially fitted method using equally-spaced grids for the type of convection-diffusion-reaction singularly perturbed one-dimensional ordinary differential equation and the following parabolic equations

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} + b(x,t)u = \epsilon \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < 1, \quad t > 0,$$

subject to $u(0,t) = u(1,t) = 0$, $t \geq 0$, $u(x,0) = u_0(x)$, $0 \leq x \leq 1$ where x and t denote the spatial coordinate and time respectively, u is the dependent variable, $a(x,t)$ is the speed and $f(x,t) - b(x,t)u$ is the reaction term. First, the time derivative is discretised by using the implicit backward Euler method in the method provided here, and then the coefficients of the resulting differential equations are frozen at each step. After this, the resulting convection-diffusion-reaction differential operator's analytical solution is obtained. For steady, constant coefficient convection-diffusion-reaction equations, this solution is exponential and accurate. For $b(x,t) = 0$, this approach agrees with exponentially fitted methods based on a constant flux density. The author showed that the exponentially-fitted approach is more reliable and has a higher order of convergence than boundary resolving upwind finite-difference schemes on piecewise-uniform meshes for the five examples considered by him (three for parabolic and two for ordinary differential equations with turning point).

In the code TOM by Mazzia and Trigiante [88], mesh selection strategies based on conditioning developed by Brugnano and Trigiante [19] have been successfully used to solve the singular perturbation problem with turning points. They implemented a strategy for hybrid mesh selection based on the estimation of two parameters that define the continuous problem conditioning, as well as a standard local truncation error calculation. In the well recognized TWPBVP code, Cash and Mazzia [21] have introduced a similar strategy. It is found that the modified code is substantially more efficient than the original code and also provides an estimation for the conditioning of the problem automatically.

A type of streamline diffusion finite element method (SDFEM) was used by Chen et al. [22] for a class of the following one-dimensional singularly perturbed problem with a boundary turning point

$$-\epsilon u'' - b(x)u' = f(x), \quad u(0) = u(1) = 0, \quad b(x) > 0, \quad x \in (0,1).$$

The authors addressed specifically the case $b(x) = x^p$, $p > 0$ but argued that it is possible to apply the same analysis to a more general case. Refined estimates of the discrete Green's function and the consistency error are used to determine uniform effects of stability and optimality. Two types of layer adapted grids, Shishkin-type grid and Bakhvalov-type grid, are proven to converge with SDFEM. It is proved that if u is the solution to the above problem, u_I is the nodal interpolation and u_h is the finite element approximation for u, then

$$\|u - u_I\|_{L^\infty(I)} \leq \begin{cases} CN^{-2}(\ln N)^{2/(p+1)}, & 1 \leq i \leq \frac{N+1}{2}, \\ CN^{-2}, & \frac{N+1}{2} + 1 \leq i \leq N+1. \end{cases}$$

on a Shishkin-type mesh. The obtained

$$\|u - u_h\|_\infty \leq CN^{-2}(\ln N)^{(p+3)/(p+1)},$$

by using triangle inequality $\|u - u_h\|_\infty \leq CM \|u - u_I\|_\infty$, whereas they got

$$\begin{aligned} &\|u - u_I\|_\infty \leq CN^{-2}, \\ &\|u - u_h\|_\infty \leq C\left(|\ln \epsilon| + \ln N\right) N^{-2}, \end{aligned}$$

for Bakhvalov grid. Thus, they obtained the following stability result on a general class of grid by incorporating the above results:

$$\|u - u_h\|_\infty \leq C\,|\ln \epsilon|\, \inf_{\nu \in V^h} \|u - v_h\|_\infty \ .$$

Since the logarithmic growth of ϵ is slow, they stated that almost second-order schemes are expected if the grid is correctly adapted, which is not easy to obtain using traditional finite-difference methods. The authors have pointed out that, compared to the traditional approach to finite-difference, the application of this uniform stability can provide an error estimate for a broad class of layer-adapted grids with a priori or a posteriori information on second-order derivatives and can also be applied to other problems, for instance, multi-grid analysis such as solver for convection-dominated problems in maximum norm.

1.4 Summary of Results

This thesis is structured as follows: In Chapter 2, we refer to a class of singularly perturbed two-point boundary value problems with a multiple-turning point and discuss some properties of the continuous problem. Specifically, we establish a minimum principle, an estimate of stability, and bounds on the solution and its derivatives. Also, we initiate a piecewise uniform graded Shishkin mesh (S-mesh) discretisation for developing numerical B-spline collocation approximation to the problem. In addition, the stability of the proposed scheme and ϵ-uniform convergence of the B-spline collocation method is given. In Chapter 3, numerical examples demonstrating the accuracy of the proposed method are discussed and compared with the existing methods. Finally, the summary of the main conclusions is given at the end of the thesis in Chapter 4.

2 Singularly Perturbed Multiple Turning Point Problem

2.1 Continuous Problem

The numerical methods for approximating the solution of singularly perturbed two-point boundary value problems for ordinary differential equations, as stated in Chapter 1, are important because these problems occur frequently in the fields of engineering and science. The influence of a small parameter in a differential equation or a boundary condition necessitates extra caution when using effective computational techniques to solve these problems. The singularly perturbed turning point problem is attractive to both applied and pure mathematicians since the solution exhibits some interesting behaviour such as boundary layer, interior layer, and resonance phenomena [14]. In particular, singularly perturbed turning point problems received much attention in the literature due to the complexity involved in finding uniformly valid asymptotic expansions, unlike non-turning problems. The problems with interior turning points represent one-dimensional versions of stationary convection-diffusion problems with a dominant convective term and a speed field that changes its sign in the catch basin. In contrast, boundary turning point problems arise in geophysics and in modeling thermal boundary layers in laminar flow [48, 115]. If one allows for higher orders of the velocity distribution, then the boundary turning point becomes multiple [128].

Consider the following class of singularly perturbed two-point boundary value problem with a multiple-turning point

$$\begin{aligned} &L_{x,\epsilon}u \equiv \epsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \\ &u(0) = U_0, \quad u(1) = U_1, \end{aligned} \tag{2.1.1}$$

where ϵ is a small perturbation parameter satisfying $0 < \epsilon \ll 1$, U_0 and U_1 are given constants and the coefficient functions $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions. The turning point is simple, if $a(x)$ vanishes at $x = 0$ and is called a multiple turning points, if not only $a(x)$ but its first derivative as well vanishes at $x = 0$. Multiple turning point problems become different from the simple turning point problems in the sense that previous theoretical results cannot be derived as they require $\lambda_i = b(x_i)/a'(x_i) < 0$ [59]. To ensure that the solution of equation (2.1.1) has multiple boundary turning point, we impose the following restriction

$$a(x) = x^p c(x), \quad p \geq 2. \tag{2.1.2}$$

Moreover, there exist a positive constant α, such that

$$c(x) \geq \alpha > 0, \quad x \in \bar{\Omega} = [0, 1]. \tag{2.1.3}$$

In order that the solution of Eq. (2.1.1) satisfies a comparison principle [28], we require that

$$b(x) \geq 0, \quad b(0) > \beta > 0. \tag{2.1.4}$$

Under these assumptions (2.1.2) - (2.1.4), the turning point problem (2.1.1) possesses a unique solution bounded uniformly in ϵ. Classical numerical methods on uniform meshes are known to be inadequate for problems with boundary layers. It is of theoretical and practical interest to deliberate numerical methods for such problems, which exhibit ϵ-uniform convergence, that is, numerical methods for which there exists an N_0, independent of ϵ, such that for all $N \geq N_0$, where N is the number of mesh elements, the error constant and rate of convergence in the maximum norm are independent of ϵ. Along these lines a numerical method is said to be ϵ-uniform of the order r on the mesh $\Omega_N = \{x_i, i = 0, 1, ..., N\}$ if there exists an N_0 independent of ϵ such that for all $N \geq N_0$

$$\sup_{0<\epsilon\leq 1} \max_{\Omega_N} |u(x) - U_N(x)| < CN^{-r}, \tag{2.1.5}$$

where $u(x)$ is the solution of the differential equation, $U_N(x)$ is the numerical approximation to $u(x)$, and C and $r > 0$ are independent of ϵ and N. Shishkin meshes are one such example of graded mesh methods which are used to obtain ϵ-uniformly convergent numerical schemes.

A large number of numerical schemes for a general singular perturbation problems have been discussed in references [37, 89, 113]. In particular, it is difficult to capture the numerical behaviour of singularly perturbed turning point boundary value problems than the singularly perturbed non-turning boundary value problems because of the vanishing character of the coefficient of the convective term. Singularly perturbed turning point problems have been extensively studied by many researchers under various assumptions. Patidar [60] introduced a finite-difference numerical scheme on a non uniform grid using cubic spline. Vigo-Aguiar [97] presented a parameter uniform numerical method based on classical upwind difference scheme on a Shishkin mesh. Kadalbajoo et al. [59] proposed a B-spline collocation method using artificial viscosity on a singularly perturbed twin boundary layer simple turning point problem. Geng [43] developed singularly perturbed turning point problems based on reproducing kernel and stretching variable. Rai [111] proposed a numerical method based on El-Mistikawy Werle exponential finite-difference scheme to solve boundary value problems for singularly perturbed differential-difference equations with a turning point. Becher [11] considered a Richardson extrapolation based on Shishkin mesh discretisation to improve the accuracy of a singularly perturbed interior turning point problem with twin exponential boundary layers. Vulanovic [82] proposed an upwind finite-difference scheme for the numerical solution of semilinear convection-diffusion problems with attractive boundary turning points. They showed that the maximum nodal error is bounded by a special weighted ℓ_1- type norm with respect to the perturbation parameter on Shishkin meshes. Lin [79] constructed a numerical algorithm based on arbitrary mesh for a quasilinear singular perturbation problem with turning points.

Lemma 1. *(Minimum Principle) Let $u(x) \in C^2(\bar{\Omega})$, satisfying $u(0) \geq 0$, $u(1) \geq 0$, such that $L_{x,\epsilon}u(x) \leq 0$, $\forall x \in \Omega$. Then $u(x) \geq 0$, $\forall x \in \bar{\Omega}$.*

Proof. Let there exists a point $x^* \in \bar{\Omega}$ such that $u(x^*) = \min\limits_{x\in\bar{\Omega}} u(x)$ and on the contrary, assume that $u(x^*) < 0$. Therefore, from the given boundary conditions, $x^* \notin \{0, 1\}$. It implies from the definition of x^* that $u'(x^*) = 0$ and $u''(x^*) \geq 0$. But then

$$L_{x,\epsilon}u(x^*) \equiv \epsilon u''(x^*) + a(x^*)u'(x^*) - b(x^*)u(x^*) > 0,$$

negates the assumption. Since x^* is an arbitrary point, it follows that $u(x) \geq 0, \forall x \in \bar{\Omega}$ and hence the minimum principle. □

Lemma 2. *(Stability Estimate) If $u(x)$ is the solution of the problem* (2.1.1)*, then $\forall \epsilon > 0$ we have*

$$\|u(x)\| \leq C \left[\max\left(|U_0|, |U_1|\right) + \tfrac{\|f\|}{\beta}\right], \quad \forall x \in \bar{\Omega}.$$

Proof. Let us define the comparison functions

$$\phi^{\pm}(x) = \max\left(|U_0|, |U_1|\right) + \tfrac{\|f\|}{\beta} \pm u(x).$$

Now applying the Lemma 1 to comparison functions $\phi^{\pm}(x)$, we get the required stability estimate. Furthermore, we give bounds for $u(x)$ and its derivatives which are important for the proof of precise representation of the solution. □

Lemma 3. *Let $y(x)$ be the solution of the problem $L_{x,\epsilon}u(x) = g(x)$, $u(0) = U_0$, $u(1) = U_1$, where $g(x) \in C^3(\bar{\Omega})$ be a function such that*

$$\left|g^{(i)}(x)\right| \leq C(1 + \epsilon^{-i/2}\exp(-\mu x)), \quad i = 0(1)3, \quad x \in \bar{\Omega}, \quad \mu = \sqrt{\tfrac{\beta}{\epsilon}}$$

where $b(0) > \beta$. Then, there exist points $\xi_i \in (0, \xi_0)$, independent of ϵ, such that

$$y^{(i)}(\xi_i) \leq C, \quad i = 1, 2, 3. \tag{2.1.6}$$

Also,

$$y^{(i)}(0) \leq C\epsilon^{-i/2}, \quad i = 1, 2, 3. \tag{2.1.7}$$

Proof. By minimum principle, it follows that the solution $y(x)$ is unique and $|y(x)| \leq C, \forall x \in \bar{\Omega}$. Let us define the functions $d_i(x)$ by

$$d_i(x) = b(x) - ia'(x), \quad i = 1, 2, 3.$$

For,

$$d_i(0) = b(0) > \beta > 0, \quad i = 1, 2, 3.$$

It follows that in the neighborhood of 0, there exist a point $\xi_0 \in \Omega$, independent of ϵ, such that

$$d_i(x) > \beta > 0, \quad x \in [0, \xi_0], \quad i = 1, 2, 3.$$

Choosing point $\xi_1 \in (0, \xi_0)$ such that $y'(\xi_1) = \xi_0^{-1}[y(\xi_0) - y(0)]$. Then (2.1.6) follows for $i = 1$. Similarly, to prove for $i = 2$, we choose ξ_2 from $(0, \xi_0)$ such that

$$y''(\xi_2) = \xi_0^{-2}[y(\xi_0) - 2y(\tfrac{\xi_0}{2}) + y(0)],$$

and ξ_3 for $i = 3$, can be found analogously. We rewrite the differential equation in the form

$$\epsilon y''(x) + (a(x)y(x))' - (a'(x) + b(x))y(x) = g(x),$$

and integrate from zero to the point $x^* \in (0, \sqrt{\epsilon})$ such that

$$y'(x^*) = \tfrac{y(\sqrt{\epsilon}) - y(0)}{\sqrt{\epsilon}}.$$

It gives $|y'(x^*)| \leq C\epsilon^{-1/2}$. To prove (2.1.7) for $i = 1$, we have

$$\epsilon y'(0) = \epsilon y'(x^*) + a(x^*)y(x^*) - \int_0^{x^*} [g(x) + (a'(x) + b(x))y(x)]dx,$$

which on simplification gives,

$$|y'(0)| \leq C(\epsilon^{-1/2}\epsilon^{p/2-1} + \epsilon^{-1}\epsilon^{1/2}) \leq C\epsilon^{-1/2}.$$

Substituting the above bounds in $L_{x,\epsilon}y(x) = g(x)$ and its successive differentiation at $x = 0$, the result follows for $i = 2, 3$. □

Lemma 4. *The solution $y(x)$ and its successive derivatives corresponding to $L_{x,\epsilon}u(x) = g(x)$, $u(0) = U_0$, $u(1) = U_1$, satisfy*

$$\left|y^{(i)}(x)\right| \leq C(1 + \epsilon^{-i/2}\exp(-\mu x)), \quad i = 1, 2, 3. \tag{2.1.8}$$

Proof. Let us define the operators

$$L_{i,x,\epsilon}u(x) \equiv L_{x,\epsilon}u(x) + ia'(x)u(x) \equiv \epsilon u''(x) + a(x)u'(x) - d_i(x)u(x), \quad i = 1, 2, 3.$$

The proof follows using comparison functions. Let us define the comparison functions

$$\psi_i^{\pm}(x) = \pm y^{(i)}(x) + C_i(1 + \epsilon^{-i/2}\exp(-\mu x)), \quad i = 1, 2, 3.$$

It is easy to show that for $i = 1, 2, 3$

$$\pm L_{i,x,\epsilon}y^{(i)}(x) \leq C(1 + \epsilon^{-i/2}\exp(-\mu x)) \tag{2.1.9}$$

and

$$L_{i,x,\epsilon}(C_i(1 + \epsilon^{-i/2}\exp(-\mu x))) \leq -C_i[d_i(x) + \epsilon^{-i/2}(d_i(x) - \beta)\exp(-\mu x)]. \tag{2.1.10}$$

From (2.1.9) and (2.1.10), it can be seen that C_i can be chosen such that

$$L_{i,x,\epsilon}\psi_i^{\pm}(x) \leq 0,\ x \in [0, \xi_i], \quad i = 1, 2, 3.$$

Also,

$$\psi_i^{\pm}(0) \geq 0,\ \psi_i^{\pm}(\xi_i) \geq 0, \quad i = 1, 2, 3.$$

Then by minimum principle lemma (Lemma 1), results (2.1.8) follows for $x \in [0, \xi_*]$, where ξ_*=min$\{\xi_1,\ \xi_2,\ \xi_3\}$. We now prove the results on $[\xi_*, 1]$, i.e.,

$$\left|y^{(i)}(x)\right| \le C, \quad i = 1, 2, 3. \tag{2.1.11}$$

Let

$$\phi(x) = \int_{\xi_*}^{x} a(t)dt = \int_{\xi_*}^{x} t^p c(t)dt.$$

Then we have,

$$\epsilon \left(e^{\phi(x)/\epsilon} y'(x)\right)' = \left(g(x) + b(x)y(x)\right) e^{\phi(x)/\epsilon},$$

$$y'(x) = \left[\frac{1}{\epsilon}\int_{\xi_*}^{x} \left(g(t) + b(t)y(t)\right) e^{\phi(t)/\epsilon} dt + y'(\xi_*)\right] e^{-\phi(x)/\epsilon}.$$

Now, using the fact that $|y'(\xi_*)| \le C$ and

$$\phi(t) - \phi(x) \le \alpha \frac{t^{p+1} - x^{p+1}}{p+1} \le \alpha \xi_*^p (t - x),$$

we get (2.1.11) for $i = 1$

$$|y'(x)| \le C, \quad x \in [\xi_*, 1].$$

Similarly, plugging the bounds ($|y'(x)| \le C$, $|g'(x)| \le C$, $x \in [\xi_*, 1]$), after differentiating $L_{x,\epsilon} y(x) = g(x)$ and expressing $y''(x)$ by means of integration, we obtain (2.1.11) for $i = 2$. Analogously, it can be proved for $i = 3$. $\square$

Theorem 1. *The solution $u(x)$ of the multiple turning point problem* (2.1.1) *can be decomposed as*

$$u(x) = \lambda w(x) + v(x), \quad \forall x \in \bar{\Omega}, \tag{2.1.12}$$

where

$$w(x) = \exp(-\mu_* x), \quad \mu_* = \sqrt{\tfrac{b(0)}{\epsilon}}, \quad |\lambda| \le C,$$

and

$$\left|v^{(i)}(x)\right| \le C(1 + \epsilon^{(1-i)/2} \exp(-\mu x)), \quad i = 0, 1, 2, 3, \quad \forall x \in \bar{\Omega}. \tag{2.1.13}$$

Proof. Let the constant λ be determined by the condition

$$v(0) = -f(0)/b(0), \ i.e.,\ v'(0) = 0.$$

From Lemma 4, we have $|\lambda| \leq C$. Also, $|v'(1)| \leq C$ and the result (2.1.13) holds for $i = 0$. Substituting the decomposed form (2.1.12) in $L_{x,\epsilon}u(x) = f(x)$ and differentiating it once, we get

$$L_{x,\epsilon}v'(x) = h(x), \quad \forall x \in \bar{\Omega}, \tag{2.1.14}$$

where

$$h(x) = f'(x) - \lambda(L_{x,\epsilon}w(x))' - a'(x)v'(x) + b'(x)v(x).$$

Now, we show that

$$\left|h^{(i)}(x)\right| \leq C(1 + \epsilon^{-i/2}\exp(-\mu x)), \quad i = 0, 1, 2, \quad \forall x \in \bar{\Omega}. \tag{2.1.15}$$

Let us consider

$$(L_{x,\epsilon}w(x))' = [\mu_*(b(x) - b(0)) + a(x)\mu_*^2 - b'(x) - a'(x)\mu_*]w(x).$$

Using mean value theorem, we have $b(x) - b(0) = xb'(\zeta), \quad \zeta \in (0, x)$, we now obtain

$$|(L_{x,\epsilon}w(x))'| \leq C(1 + x^{p-1}\epsilon^{-1/2} + x\epsilon^{-1/2} + x^p\epsilon^{-1})w(x) \leq C.$$

Also,

$$|a'(x)v'(x)| \leq Cx^{p-1}(|u'(x)| + |w'(x)|) \leq C.$$

Thus, we get (2.1.15) for $i = 0$. Similarly [66] by applying Lemma 4 on (2.1.14), we obtain

$$\left|v'^{(i)}(x)\right| \leq C(1 + \epsilon^{-i/2}\exp(-\mu x)), \quad i = 0, 1, 2, \quad \forall x \in \bar{\Omega},$$

which is nothing, but (2.1.13) for $i = 1, 2, 3$. This representation shows that $u(x)$ has an $O(\sqrt{\epsilon})$ boundary layer at $x = 0$. □

2.2 B-spline Collocation Method

In this section, we consider the appropriate discretisation that will be used for developing numerical B-spline collocation approximation to the problem. As the problem (2.1.1) has a boundary layer in the vicinity of $x = 0$, we introduce a fitted piecewise uniform Shishkin mesh to overcome the oscillations as $\epsilon \to 0$ that discretises $\bar{\Omega} = [0, 1]$ with $N = \{2^m, m \geq 2\}$ mesh elements. The mesh is piecewise uniform depending on one transition point which is defined using the transition parameter $\tau = \min\left\{\frac{1}{2}, \tau_0\sqrt{\epsilon}\ln N\right\}$,where $\tau_0 \geq \frac{1}{\sqrt{\beta}}$ is a constant independent of N, ϵ and to be fixed later. The mesh is generated by setting a uniform mesh with $N/2$ mesh elements in each of $\Omega_1 = [0, \tau]$ and $\Omega_2 = [\tau, 1]$ such that $\bar{\Omega} = \Omega_1 \cup \Omega_2$. The mesh elements are given by

$$\bar{\Omega}_\tau^N = \left\{ x_i \mid x_i = \begin{cases} ih, & 0 \leq i \leq N/2 \\ \tau + (i - N/2)\, h, & N/2 + 1 \leq i \leq N \end{cases} \right\}, \tag{2.2.1}$$

where mesh spacing is

$$h = \left\{ h_i \mid h_i = \begin{cases} 2\tau/N, & 1 \leq i \leq N/2 \\ 2\,(1-\tau)\,/N, & N/2 + 1 \leq i \leq N \end{cases} \right\}.$$

Here, we describe the B-spline collocation method to obtain the approximate solution to the multiple turning point problem (2.1.1) with fitted mesh. We assume X is a linear subspace of $L_2(\bar{\Omega})$, the space of all square integrable functions defined on $\bar{\Omega}$. A cubic B-splines ϕ_i, $i = -1, 0, ..., N+1$, covers four elements and defined over the interval $[0, 1]$ as follows [110]:

$$\phi_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3, & x_{i-2} \leq x \leq x_{i-1}, \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x_{i-1} \leq x \leq x_i, \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x_i \leq x \leq x_{i+1}, \\ (x_{i+2} - x)^3, & x_{i+1} \leq x \leq x_{i+2}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.2}$$

It is easy to see that each $\phi_i(x)$ is also a piecewise cubic with knots at $\bar{\Omega}$ and $\phi_i(x) \in X$, also $\phi_i(x)$ is twice continuously differentiable $\forall x \in \Re$. Let $\Lambda = \{\phi_{-1}, \phi_0, ..., \phi_{N+1}\}$ and $\varPhi_3\left(\bar{\Omega}_\tau^N\right) = \left\{\Psi_i : \Psi_i = \sum\limits_{i=-1}^{N+1} k_i\phi_i, \quad k_i \in \Re\right\}$. The functions in Λ are linearly independent on $\bar{\Omega}$, thus $\varPhi_3\left(\bar{\Omega}_\tau^N\right)$ is $(N+3)$ dimensional.

Table 2.2.1: Cubic B-spline basis and its derivatives function values at nodal points.

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$\phi_i(x)$	0	1	4	1	0
$\phi_i'(x)$	0	$\frac{3}{h}$	0	$-\frac{3}{h}$	0
$\phi_i''(x)$	0	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	0

Let $L_{x,\epsilon}$ be a linear operator whose domain is X and whose range is also in X. Also, the nodal values and its derivatives at the nodes x_i's are given by Table 2.2.1. Let $\Phi_3\left(\bar{\Omega}_\tau^N\right)$ be an $(N+3)$ dimensional subspace of X. Now suppose the approximate solution is given by

$$U(x) = \sum_{i=-1}^{N+1} e_i \phi_i(x), \tag{2.2.3}$$

where e_i's are unknown real coefficients and $\phi_i(x)$ are cubic B-spline functions. This representation shows the variation of all contributing cubic B-splines over a single element and is useful for working out the solution inside the element. Now, we introduce two superfluous cubic B-splines, ϕ_{-1} and ϕ_{N+1} to satisfy the boundary conditions. Furthermore, it is required that the approximate solution $U(x)$ satisfies the given problem (2.2.1) at mesh points as well as boundary conditions at $x = x_0$ and $x = x_N$. Therefore, we have

$$L_{x,\epsilon} U(x_i) = f(x_i), \quad 0 \le i \le N, \tag{2.2.4}$$

and

$$U(x_0) = U_0, \quad U(x_N) = U_1. \tag{2.2.5}$$

Solving the collocation stencil (2.2.4), we obtain a system of $(N+1)$ linear equations in $(N+3)$ unknowns

$$\begin{aligned} e_{i-1}(\epsilon\phi''_{i-1}(x_i) + a_i\phi'_{i-1}(x_i) - b_i\phi_{i-1}(x_i)) + e_i(\epsilon\phi(x_i) + a_i\phi'_i(x_i) - b_i\phi_i(x_i)) \\ + e_{i+1}(\epsilon\phi''_{i+1}(x_i) + a_i\phi'_{i+1}(x_i) - b_i\phi_{i+1}(x_i)) = f_i, \quad 0 \le i \le N, \end{aligned} \tag{2.2.6}$$

where $a(x_i) = a_i$, $b(x_i) = b_i$ and $f(x_i) = f_i$. Substituting the values of B-spline functions ϕ_i and its derivatives at mesh points, we get

$$r_i^- e_{i-1} + r_i^c e_i + r_i^+ e_{i+1} = h^2 f_i, \quad 0 \le i \le N, \tag{2.2.7}$$

where $r_i^- = 6\epsilon - 3a_i h - b_i h^2$, $r_i^c = -12\epsilon - 4b_i h^2$, $r_i^+ = 6\epsilon + 3a_i h - b_i h^2$.

The given boundary conditions become

$$e_{-1} + 4e_0 + e_1 = U_0, \tag{2.2.8}$$

and

$$e_{N-1} + 4e_N + e_{N+1} = U_1. \tag{2.2.9}$$

Thus, we obtain a $(N+3) \times (N+3)$ system in $(N+3)$ unknowns $\{e_{-1}, e_0, ..., e_{N+1}\}$ by using equations (2.2.7) - (2.2.9). Eliminating e_{-1} from first equation of (2.2.7) and from (2.2.8), we find

$$(r_0^c - 4r_0^-)e_0 + (r_0^+ - r_0^-)e_1 = f_0h^2 - U_0r_0^-. \tag{2.2.10}$$

Similarly, eliminating e_{N+1} from the last equation of (2.2.7) and from (2.2.9), we get

$$(r_N^- - r_N^+)e_{N-1} + (r_N^c - 4r_N^+)e_N = f_Nh^2 - U_1r_N^+. \tag{2.2.11}$$

Thus, putting off e_{-1} and e_{N+1} leads to a system of $(N+1)$ linear equations in $(N+1)$ unknowns

$$R_h e = q, \tag{2.2.12}$$

where $e = (e_0, e_1, ..., e_N)^T$ are the unknown real coefficients with right hand side $q = (q_0, q_1, ..., q_N)^T$. The coefficient matrix is given by

$$R_h = \begin{bmatrix} (r_0^c - 4r_0^-) & (r_0^+ - r_0^-) & 0 & 0 & \dots & 0 \\ r_1^- & r_1^c & r_1^+ & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & r_i^- & r_i^c & r_i^+ & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & r_{N-1}^- & r_{N-1}^c & r_{N-1}^+ \\ 0 & \dots & 0 & 0 & (r_N^- - r_N^+) & (r_N^c - 4r_N^+) \end{bmatrix}.$$

The elements of the column vector q are

$$q_i = \begin{cases} f_0h^2 - U_0r_0^-, & i = 0, \\ f_ih^2, & i = 1(1)N-1, \\ f_Nh^2 - U_1r_N^+, & i = N. \end{cases}$$

It is observed that collocation matrix R_h is strictly diagonally dominant and hence non-singular. Solving the above matrix system gives the values of $e = (e_0, e_1, ..., e_N)^T$ which when coupled with the boundary conditions (2.2.8) and (2.2.9), we obtain the unknowns e_{-1} and e_{N+1}. Hence, the method of collocation using a basis of cubic B-splines when applied to problem 2.1.1 has a unique solution, $U(x)$, given by (2.2.3).

2.3 Stability and Convergence Analysis

In this section, we analyse the stability and convergence estimate in the maximum norm. The collocation method for solving $R_h e = q$ is said to be stable, if there exist positive constants C_1,C_2 and C_3 such that the perturbed system has a unique solution for $\|\delta R_h\| \leq C_3$ and

$$\|e - \widetilde{e}\| \leq \left(C_1 \|\delta R_h\| \|e\| + C_2 \|\delta_q\|\right).$$

Suppose a small error δR_h and δ_q has been made in the calculations of R_h and q respectively. Let $\widetilde{e}$ be the solution of the perturbed system

$$(R_h + \delta R_h)\,\widetilde{e} = q + \delta q. \tag{2.3.1}$$

We have seen that R_h is strictly diagonally dominant. Therefore, by a result in [123], for a sufficiently small value of h, we have

$$\left\|R_h^{-1}\right\| \leq \frac{C}{h^2} = \kappa. \tag{2.3.2}$$

Choose a positive constant $r < \frac{1}{2\kappa}$. Then whenever $\|\delta R_h\| < r$, (2.3.1) has a unique solution, for

$$\left\|(R_h + \delta R_h)^{-1}\right\| = \left\|\left(I + R_h^{-1}\delta R_h\right)^{-1} R_h^{-1}\right\| \leq 2\kappa,$$

because

$$\left\|R_h^{-1}\delta R_h\right\| \leq \left\|R_h^{-1}\right\| \|\delta R_h\| < \frac{1}{2}.$$

Since

$$(R_h + \delta R_h)\,(e - \widetilde{e}) = \delta R_h e - \delta q,$$

it follows that

$$\|e - \widetilde{e}\| \leq 2\kappa \left(\|\delta R_h\| \|e\| + \|\delta q\|\right), \tag{2.3.3}$$

which ensures the stability of the collocation system.

Lemma 5. *The third degree B-splines* $\Lambda = \{\phi_{-1}, \phi_0, ..., \phi_{N+1}\}$ *satisfy the following in-*

equality

$$\sum_{i=-1}^{N+1} |\phi_i(x)| \le 10, \quad 0 \le x \le 1.$$

Proof. The proof easily follows by the definition of the third degree B-spline given by (2.2.2).

Error estimate $(U-u)$ is given in the following theorem, where $U(x)$ is the cubic B-spline collocation approximate of the exact solution. □

Theorem 2. *Let $u(x)$ be the solution of the problem and $U(x)$ be the corresponding collocation approximation from the space of cubic splines $\Phi_3\left(\bar{\Omega}_\tau^N\right)$, then for sufficiently small values of h and ϵ, we have*

$$|U(x)| \le C, \quad x \in \bar{\Omega},$$

where C is a generic positive constant.

Proof. For sufficiently small h and ϵ, it follows from the result in (2.3.2) and (2.2.12),

$$\|e\| \le \left\|R_h^{-1}\right\| \|q\| \le C.$$

Using Lemma 5, and boundedness of coefficients e_{-1} and e_{N+1}, by boundary conditions (2.2.8) and (2.2.9), we have

$$|U(x)| = \left|\sum_{i=-1}^{N+1} e_i\phi_i(x)\right| \le \sum_{i=-1}^{N+1} |e_i|\,|\phi_i(x)| \le C, \quad x \in \bar{\Omega},$$

Now, ϵ-uniform error estimate is given by the following theorem. □

Theorem 3. *Let $u(x)$ be sufficiently smooth solution of the problem and $U(x)$ be the corresponding collocation approximation from the space of cubic splines $\Phi_3\left(\bar{\Omega}_\tau^N\right)$ on a Shishkin mesh. If $f \in C^2[0,1]$, then the ϵ-uniform error estimate is given by*

$$\sup_{0<\epsilon\ll 1} \|(U-u)(x)\| \le CN^{-2}(\ln N)^2$$

where C is a positive constant independent of ϵ and N.

Proof. Let $Y(x)$ be the unique spline interpolate from $\Phi_3\left(\bar{\Omega}_\tau^N\right)$ to the exact solution $u(x)$ of the boundary value problem given by

$$Y(x) = \sum_{i=-1}^{N+1} l_i\phi_i(x). \tag{2.3.4}$$

If $f(x) \in C^2[0,1]$, then $u(x) \in C^4[0,1]$ and it follows from Hall error estimates [47, 110] that

$$\left|D^j(u(x)-Y(x))\right| \leq \omega_j \left|u^{(4)}(x)\right| h^{4-j}, \quad j=0,1,2, \tag{2.3.5}$$

where ω_j's are constants independent of h and N. Hence, from estimate (2.3.5) it follows that

$$\left|L_{x,\epsilon}u(x_i) - L_{x,\epsilon}Y(x_i)\right| \leq C\left(\epsilon\omega_2 + \omega_1\left\|a\right\|h + \omega_0\left\|b\right\|h^2\right)\left|u^{(4)}(x)\right|h^2. \tag{2.3.6}$$

Assume that

$$L_{x,\epsilon}Y(x_i) = \widetilde{f}(x_i), 0 \leq i \leq N,$$

which produces the linear algebraic matrix $R_h l = \widetilde{q}$, where $l = (l_0, l_1, ..., l_N)^T$ and $\widetilde{q} = (\widetilde{q}_0, \widetilde{q}_1, ..., \widetilde{q}_N)^T$ are column vectors. Now,

$$(e-l) = R_h^{-1}\left(q - \widetilde{q}\right), \tag{2.3.7}$$

where

$$q-\widetilde{q} = \left(h^2\left(f(x_0)-\widetilde{f}(x_0)\right), h^2\left(f(x_1)-\widetilde{f}(x_1)\right), \ldots, h^2\left(f(x_N)-\widetilde{f}(x_N)\right)\right)^T. \tag{2.3.8}$$

We consider different cases based on Shishkin mesh characterization:

Case 1. When $\tau = \frac{1}{2}$, it implies that mesh is uniform with spacing $h = \frac{1}{N}$ and $\min\{\tau_0\sqrt{\epsilon}\ln N\} \geq \frac{1}{2}$. Using estimate (2.3.6), $\epsilon^{-1} \leq (2\tau_0 \ln N)^2$, Lemma 4 and the argument that $\epsilon \ll 1$ and $\epsilon^{-k/2}\exp(-\mu x) \to 0$ as $\epsilon \to 0\, \forall x \in \Omega$, $k \in \mathbb{I}^+$, we get

$$\left|L_{x,\epsilon}u(x_i) - L_{x,\epsilon}Y(x_i)\right| \leq CN^{-2}(\ln N)^2. \tag{2.3.9}$$

Case 2. When $\tau < \frac{1}{2}$, it implies that mesh is piecewise uniform with spacing $h = \frac{2\tau}{N}$ for $\{x_i\}_{i=0}^{\frac{N}{2}}$ in Ω_1 and $h = \frac{2(1-\tau)}{N}$ for $\{x_i\}_{i=\frac{N}{2}+1}^{N}$ in Ω_2 where $\tau = \tau_0\sqrt{\epsilon}\ln N$. For $\{x_i\}_{i=\frac{N}{2}+1}^{N}$ in Ω_2, we have $\left|y^{(i)}(x\right| \leq C$ from (2.1.11), it follows from (2.3.6) that

$$\left|L_{x,\epsilon}u(x_i) - L_{x,\epsilon}Y(x_i)\right| \leq CN^{-2}. \tag{2.3.10}$$

Furthermore, if $\{x_i\}_{i=0}^{\frac{N}{2}}$ in Ω_1, we have $h = 2\tau_0\sqrt{\epsilon}N^{-1}\ln N$, thus it follows from Lemma 4, (2.3.6) and the fact that $\epsilon \ll 1$ and $\epsilon^{-k/2}\exp(-\mu x) \to 0$ as $\epsilon \to 0\, \forall x \in \Omega$, $k \in \mathbb{I}^+$,we get

$$\left|L_{x,\epsilon}u(x_i) - L_{x,\epsilon}Y(x_i)\right| \leq CN^{-2}(\ln N)^2. \tag{2.3.11}$$

By incorporating (2.3.9), (2.3.10) and (2.3.11), we have

$$|L_{x,\epsilon}u(x_i) - L_{x,\epsilon}Y(x_i)| \leq CN^{-2}(\ln N)^2. \tag{2.3.12}$$

Hence,

$$|L_{x,\epsilon}U(x_i) - L_{x,\epsilon}Y(x_i)| = |f(x_i) - L_{x,\epsilon}Y(x_i)| = |L_{x,\epsilon}u(x_i) - L_{x,\epsilon}Y(x_i)| \leq CN^{-2}(\ln N)^2. \tag{2.3.13}$$

(2.3.8) and (2.3.13) imply that

$$\|q - \widetilde{q}\| \leq CN^{-4}(\ln N)^2. \tag{2.3.14}$$

From estimates (2.3.2), (2.3.7) and (2.3.14), we get

$$(e_i - l_i) \leq CN^{-2}(\ln N)^2, \quad 0 \leq i \leq N. \tag{2.3.15}$$

Using (2.3.15) and boundedness of coefficients e_{-1} and e_{N+1}, by boundary conditions (2.2.8) and (2.2.9), we have

$$(e_i - l_i) \leq CN^{-2}(\ln N)^2, \quad -1 \leq i \leq N+1. \tag{2.3.16}$$

Thus, using above estimate (2.3.16) and Lemma 5, we obtain

$$\|U(x) - Y(x)\| \leq \max_{-1 \leq i \leq N+1} |e_i - l_i| \sum_{i=-1}^{N+1} |\phi_i(x)| \leq CN^{-2}(\ln N)^2. \tag{2.3.17}$$

Finally, using triangle inequality, Hall estimate (2.3.5) and (2.3.17), we have

$$\sup_{0<\epsilon\ll 1} \|(U-u)(x)\| \leq CN^{-2}(\ln N)^2.$$

□

2.4 Summary of Results

In this chapter, explicit bounds for the solution of the turning point problem and its derivatives are derived. We prove that the operator $L_{x,\epsilon}$ as defined in (2.1.1) satisfies a minimum principle. Then we state a stability estimate for the solution of (2.1.1). Also, a numerical method based on B-spline collocation is presented to solve the singularly perturbed multiple turning point problem. These methods can be closely related to Galerkin methods, hence to finite-element methods, as they are much easier and more efficient for computing. B-splines are used because they yield results of higher accuracy as compared with those of polynomial interpolation. The analysis has been given for the stability and convergence of the B-spline collocation method wherein the collocation method gives a system of linear equations.

3 Results and Discussion

3.1 Numerical Results

In this section, we consider some known numerical examples for singularly perturbed multiple turning point problems to demonstrate the efficiency of the proposed method presented in this paper.

Example 1. [128].We consider the example

$$\epsilon u''(x) + x^p u'(x) - u(x) = f(x), \quad x \in (0,1) \ ,$$

$$u(0) = 2, \quad u(1) \approx e,$$

whose exact solution is given by

$$u(x) = e^{\frac{-x}{\sqrt{\epsilon}}} + e^x,$$

from which we determine $f(x)$. Since the problem has an analytical solution, therefore, for every ϵ the computed maximum pointwise errors are estimated by

$$E_\infty^{N,\epsilon} = \|u - U_{N,\epsilon}\|_\infty = \max_{0 \le i \le N} |u(x_i) - U_{N,\epsilon}(x_i)| \, ,$$

where u and $U_{N,\epsilon}$ are the exact and computed solutions respectively. Also, the ϵ-uniform maximum point-wise error is computed as

$$E_\infty^N = \max_\epsilon E_\infty^{N,\epsilon}.$$

Table 3.1.1: Maximum pointwise errors $E_{\infty}^{N,\epsilon}$ for Example 1 with uniform mesh.

$p, \epsilon \downarrow$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$2, 2^{-16}$	$1.106E+00$	$1.021E-01$	$1.416E-01$	$6.291E-02$	$1.684E-02$	$3.914E-03$	$9.618E-04$
$2, 2^{-20}$	$2.014E+01$	$2.501E+00$	$2.373E-01$	$1.956E-01$	$1.496E-01$	$6.338E-02$	$1.687E-02$
$2, 2^{-25}$	$6.499E+02$	$8.394E+01$	$1.051E+01$	$1.243E+00$	$1.120E-01$	$2.294E-01$	$1.944E-01$
E_{∞}^{N}	$6.499E+02$	$8.394E+01$	$1.051E+01$	$1.243E+00$	$1.496E-01$	$2.294E-01$	$1.944E-01$
$3, 2^{-16}$	$1.850E-01$	$2.241E-01$	$1.521E-01$	$6.353E-02$	$1.688E-02$	$3.922E-03$	$9.635E-04$
$3, 2^{-20}$	$2.517E+00$	$1.892E-01$	$2.527E-01$	$2.277E-01$	$1.522E-01$	$6.354E-02$	$1.688E-02$
$3, 2^{-25}$	$8.321E+01$	$0.521E+01$	$2.802E-01$	$2.570E-01$	$2.620E-01$	$2.470E-01$	$1.961E-01$
E_{∞}^{N}	$8.321E+01$	$0.521E+01$	$2.802E-01$	$2.570E-01$	$2.620E-01$	$2.470E-01$	$1.961E-01$

Table 3.1.2: Maximum pointwise errors $E_{\infty}^{N,\epsilon}$ for Example 1 with fitted Shishkin mesh.

$p, \epsilon \downarrow$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$2, 2^{-16}$	$1.523E-02$	$8.871E-03$	$5.211E-03$	$2.153E-03$	$6.626E-04$	$3.168E-04$	$1.466E-04$
$2, 2^{-20}$	$1.495E-02$	$8.236E-03$	$4.700E-03$	$2.750E-03$	$1.407E-03$	$5.032E-04$	$1.750E-04$
$2, 2^{-25}$	$1.488E-02$	$8.093E-03$	$4.536E-03$	$2.491E-03$	$1.362E-03$	$7.482E-04$	$3.805E-04$
E_{∞}^{N}	$1.523E-02$	$8.871E-03$	$5.211E-03$	$2.750E-03$	$1.407E-03$	$7.482E-04$	$3.805E-04$
$3, 2^{-16}$	$1.455E-02$	$7.506E-03$	$3.824E-03$	$1.356E-03$	$5.881E-04$	$5.993E-04$	$2.667E-04$
$3, 2^{-20}$	$1.462E-02$	$7.865E-03$	$4.408E-03$	$2.338E-03$	$1.143E-03$	$3.931E-04$	$2.120E-04$
$3, 2^{-25}$	$1.462E-02$	$7.951E-02$	$4.460E-03$	$2.447E-03$	$1.311E-03$	$6.853E-04$	$3.403E-04$
E_{∞}^{N}	$1.462E-02$	$7.951E-02$	$4.460E-03$	$2.447E-03$	$1.311E-03$	$6.853E-04$	$3.403E-04$

Table 3.1.3: Comparison of maximum errors $\left(E_{\infty}^{N,\epsilon}\right)$ of the present method with finite-difference method (FDM), when applied to Example 1 for $p = 2$ and different values of ϵ, N.

	FDM [128]			Present Method		
$\epsilon \downarrow$	$N = 50$	$N = 100$	$N = 200$	$N = 50$	$N = 100$	$N = 200$
10^{-6}	$3.190E-02$	$1.55E-02$	$7.78E-03$	$5.754E-03$	$3.329E-03$	$1.834E-03$
10^{-12}	$3.65E-02$	$1.83E-02$	$9.11E-03$	$5.543E-03$	$3.059E-03$	$1.660E-03$

Example 2. Now we consider the following problem,

$$\epsilon u''(x) + \tfrac{1}{2}x^p u'(x) - u(x) = f(x), \quad x \in (0, 1),$$

$$u(0) = 1, \quad u(1) \approx 1,$$

with the exact solution

$$u(x) = e^{\frac{-x}{\sqrt{\epsilon}}} + x^2,$$

from which we determine $f(x)$.

The estimated maximum pointwise errors for test Examples 1, 2 and 3 are presented in Tables 3.1.1, 3.1.2 and 3.1.4 to 3.1.7 with uniform mesh and fitted Shishkin mesh. The results are presented for two values of $p = 2, 3$, although it can be extended for higher values.

Table 3.1.4: Maximum pointwise errors $E_{\infty}^{N,\epsilon}$ for Example 2 with uniform mesh.

$p, \epsilon \downarrow$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$2, 2^{-18}$	$2.501E+00$	$2.373E-01$	$1.956E-01$	$1.496E-01$	$6.338E-02$	$1.687E-02$	$3.920E-03$
$2, 2^{-20}$	$1.038E+01$	$1.227E+00$	$1.109E-01$	$2.117E-01$	$1.509E-01$	$6.346E-02$	$1.687E-02$
$2, 2^{-25}$	$3.361E+02$	$4.235E+01$	$5.229E+00$	$5.856E-01$	$1.871E-01$	$2.382E-01$	$1.952E-01$
E_{∞}^{N}	$3.361E+02$	$4.235E+01$	$5.229E+00$	$5.856E-01$	$1.871E-01$	$2.382E-01$	$1.952E-01$
$3, 2^{-18}$	$2.756E-01$	$2.482E-01$	$2.275E-01$	$1.522E-01$	$6.354E-02$	$1.688E-02$	$3.922E-03$
$3, 2^{-20}$	$1.236E+00$	$2.272E-01$	$2.550E-01$	$2.278E-01$	$1.523E-01$	$6.354E-02$	$1.688E-02$
$3, 2^{-25}$	$4.198E+01$	$2.574E+00$	$1.904E-01$	$2.618E-01$	$2.623E-01$	$2.470E-01$	$1.961E-01$
E_{∞}^{N}	$4.198E+01$	$2.574E+00$	$2.550E-01$	$2.618E-01$	$2.623E-01$	$2.470E-01$	$1.961E-01$

Table 3.1.5: Maximum pointwise errors $E_{\infty}^{N,\epsilon}$ for Example 2 with fitted Shishkin mesh.

$p, \epsilon \downarrow$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$2, 2^{-18}$	$6.644E-03$	$2.347E-03$	$8.421E-04$	$2.856E-04$	$9.322E-05$	$2.630E-05$	$1.672E-05$
$2, 2^{-20}$	$6.719E-03$	$2.347E-03$	$8.422E-04$	$2.857E-04$	$9.324E-05$	$2.949E-05$	$8.117E-06$
$2, 2^{-25}$	$6.762E-03$	$2.347E-03$	$8.422E-04$	$2.768E-04$	$8.855E-05$	$2.862E-05$	$9.102E-06$
E_{∞}^{N}	$6.769E-03$	$2.347E-03$	$8.422E-04$	$2.857E-04$	$9.324E-05$	$2.949E-05$	$1.672E-05$
$3, 2^{-18}$	$6.771E-03$	$2.348E-03$	$8.424E-04$	$2.858E-04$	$9.326E-05$	$2.631E-05$	$1.811E-05$
$3, 2^{-20}$	$6.767E-03$	$2.347E-03$	$8.423E-04$	$2.857E-04$	$9.326E-05$	$2.950E-05$	$8.119E-06$
$3, 2^{-25}$	$6.762E-03$	$2.347E-03$	$8.422E-04$	$2.786E-04$	$8.928E-05$	$2.891E-05$	$9.103E-06$
E_{∞}^{N}	$6.771E-03$	$2.348E-03$	$8.424E-04$	$2.858E-04$	$9.326E-05$	$2.950E-05$	$1.811E-05$

Example 3. Finally, we consider the following problem,

$$\epsilon u''(x) + x^p u'(x) - (\tfrac{1}{2} - x)^2 u(x) = -e^x, \quad x \in (0, 1),$$

$$u(0) = \quad u(1) = 0,$$

which do not have a closed form of the exact solution.

Since the given problem does not posses analytical solution, therefore, we use the double mesh principle [26] to estimate the maximum pointwise error as follows

$$E^{N,\epsilon} = \max_{0\le i\le N} \left|U_{N,\epsilon}(x_i) - U_{2N,\epsilon}(x_i)\right|,$$

where $U_{2N,\epsilon}(x_i)$ is the solution obtained on a mesh, containing the same number N of mesh points used to compute $U_{N,\epsilon}(x_i)$ and N more mesh points are added by selecting the mid points of all x_i's, i.e., $x_{i+1/2} = (x_i + x_{i+1})/2$ for $i = 1, 2, ..., N-1$. Also, the ϵ-uniform maximum point-wise error is computed as

$$E^N = \max_{\epsilon} E^{N,\epsilon}.$$

Table 3.1.6: Maximum pointwise errors $E^{N,\epsilon}$ for Example 3 with uniform mesh.

$p,\epsilon\downarrow$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
$2,2^{-15}$	$2.371E+00$	$2.899E-01$	$1.111E-01$	$2.466E-02$	$6.042E-03$	$1.497E-03$	$3.741E-04$
$2,2^{-20}$	$1.453E+02$	$1.449E+01$	$7.644E-01$	$4.888E-01$	$2.041E-01$	$5.217E-02$	$1.189E-02$
$2,2^{-25}$	$4.752E+03$	$5.309E+02$	$4.201E+01$	$4.831E+00$	$5.487E-01$	$7.287E-01$	$3.541E-01$
E^N	$4.752E+03$	$5.309E+02$	$4.201E+01$	$4.831E+00$	$5.487E-01$	$7.287E-01$	$3.541E-01$
$3,2^{-15}$	$8.752E-01$	$3.942E-01$	$1.156E-01$	$2.470E-02$	$6.172E-03$	$1.526E-03$	$3.809E-04$
$3,2^{-20}$	$3.205E+01$	$1.042E+00$	$1.023E+00$	$5.707E-01$	$2.086E-01$	$5.241E-02$	$1.195E-02$
$3,2^{-25}$	$1.108E+03$	$3.127E+01$	$1.363E+00$	$1.292E+00$	$1.151E+00$	$7.839E-01$	$3.576E-01$
E^N	$1.108E+03$	$3.127E+01$	$1.363E+00$	$1.292E+00$	$1.151E+00$	$7.839E-01$	$3.576E-01$

Table 3.1.7: Maximum pointwise errors $E^{N,\epsilon}$ for Example 3 with fitted Shishkin mesh.

$p,\epsilon\downarrow$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
$2,2^{-15}$	$2.368E-01$	$9.143E-02$	$1.044E-02$	$1.760E-03$	$1.022E-03$	$7.168E-04$	$2.537E-04$
$2,2^{-20}$	$4.210E-01$	$1.973E-01$	$7.935E-02$	$2.234E-02$	$6.392E-03$	$1.718E-03$	$4.453E-04$
$2,2^{-25}$	$7.411E-01$	$4.996E-01$	$1.924E-01$	$9.745E-02$	$4.971E-02$	$2.421E-02$	$8.621E-03$
E^N	$7.411E-01$	$4.996E-01$	$1.924E-01$	$9.745E-02$	$4.971E-02$	$2.421E-02$	$8.621E-03$
$3,2^{-15}$	$4.857E-01$	$1.111E-01$	$1.126E-02$	$2.980E-03$	$1.610E-03$	$1.185E-03$	$3.491E-04$
$3,2^{-20}$	$6.939E-01$	$1.087E-01$	$1.580E-01$	$8.597E-02$	$2.120E-02$	$5.753E-03$	$1.474E-03$
$3,2^{-25}$	$1.143E+00$	$5.601E-01$	$7.327E-02$	$4.956E-02$	$3.812E-02$	$2.425E-02$	$1.138E-02$
E^N	$1.143E+00$	$5.601E-01$	$1.580E-01$	$8.597E-02$	$3.812E-02$	$2.425E-02$	$1.138E-02$

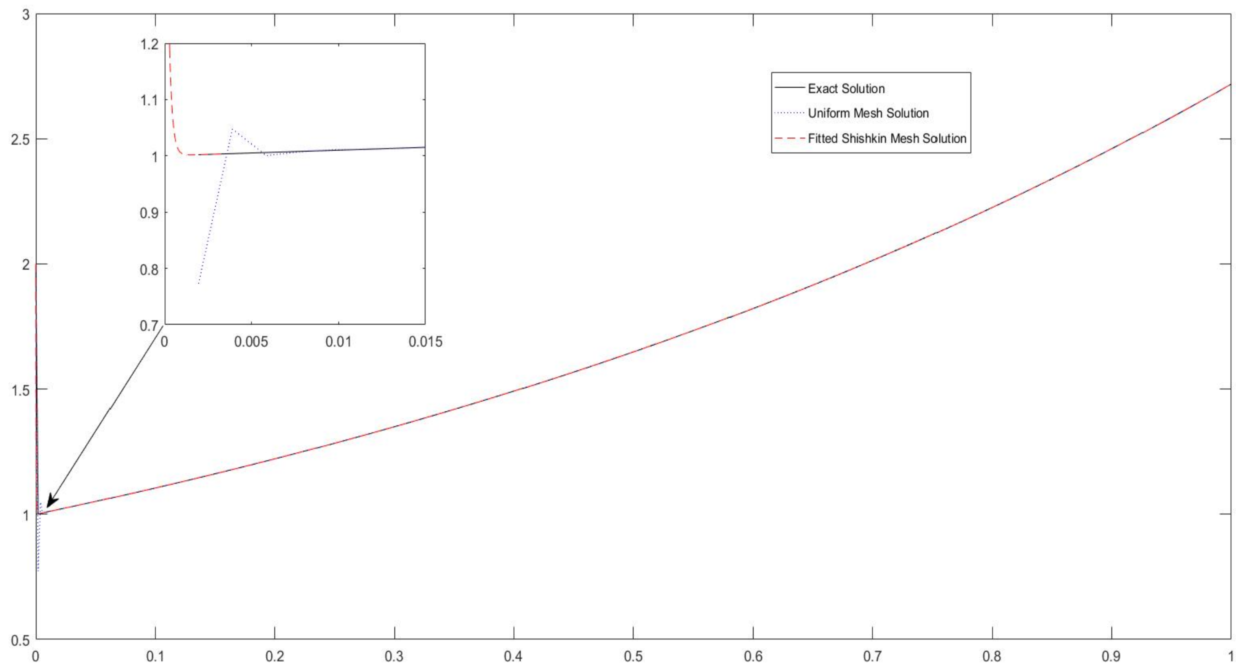

Figure 3.1.1: Exact and proposed method numerical solutions for Example 1 for $\epsilon = 2^{-25}$, $p = 2$ and $N = 512$ with fitted Shishkin mesh.

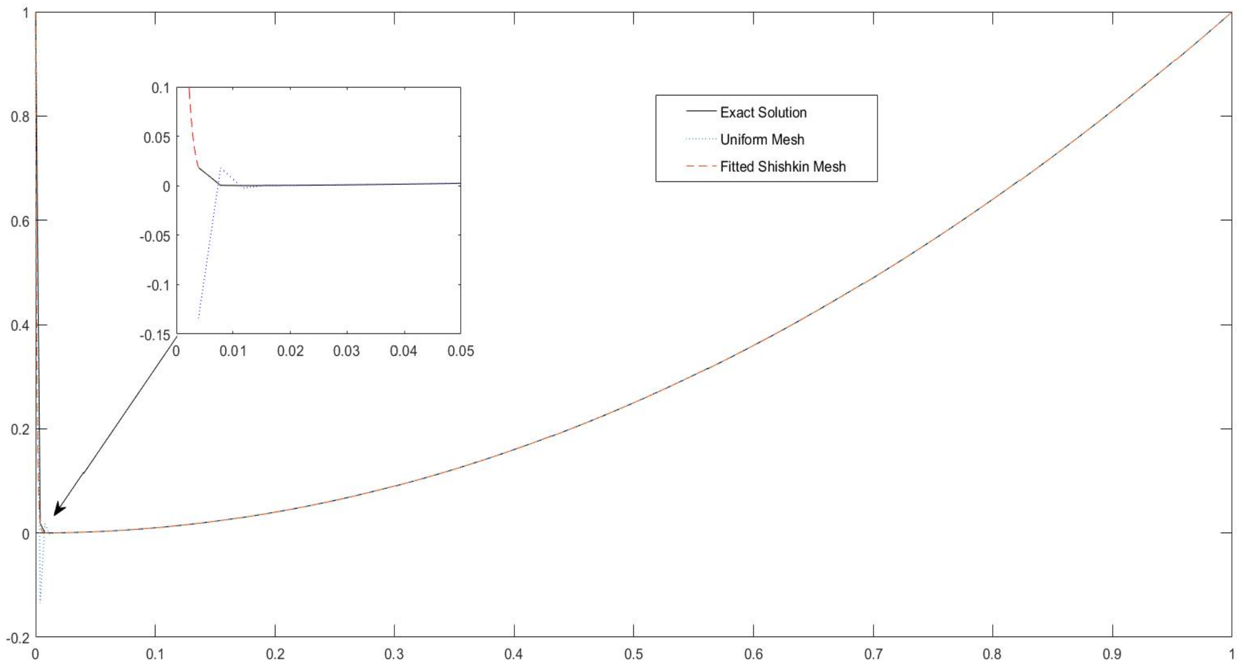

Figure 3.1.2: Exact and proposed method numerical solutions for Example 2 for $\epsilon = 2^{-20}$, $p = 3$ and $N = 256$ with fitted Shishkin mesh.

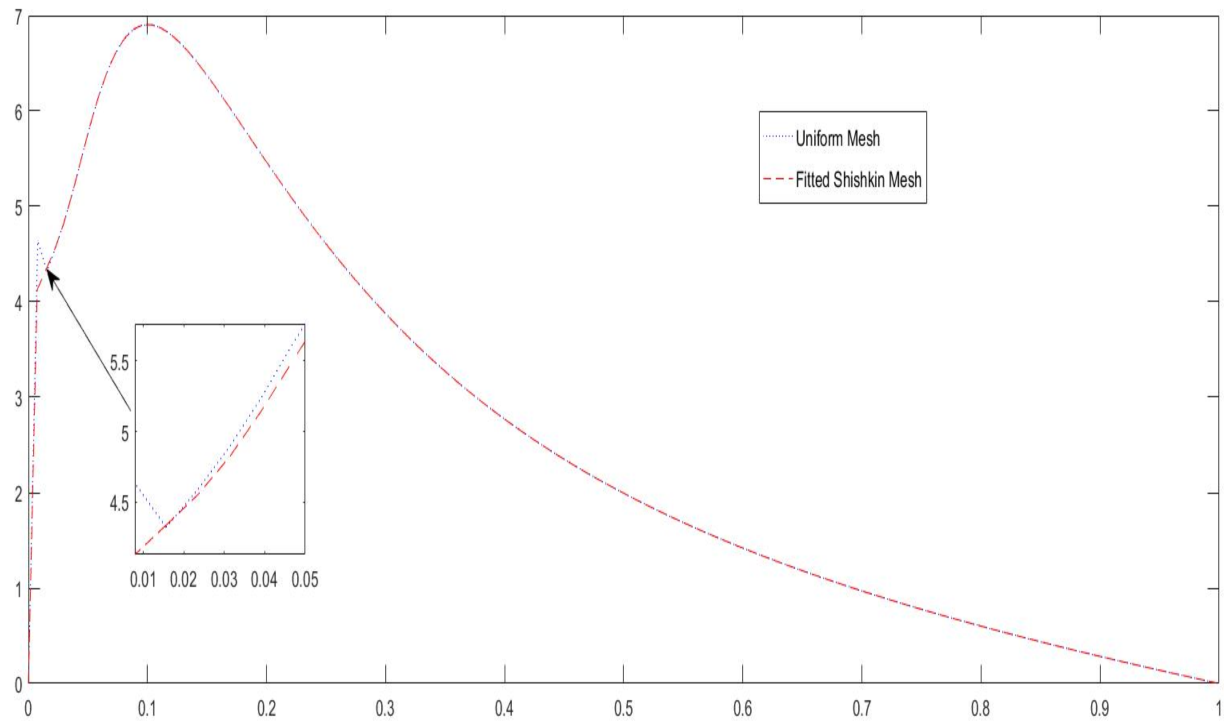

Figure 3.1.3: Uniform and fitted numerical solutions for Example 3 for $\epsilon = 2^{-20}$, $p = 2$ and $N = 128$.

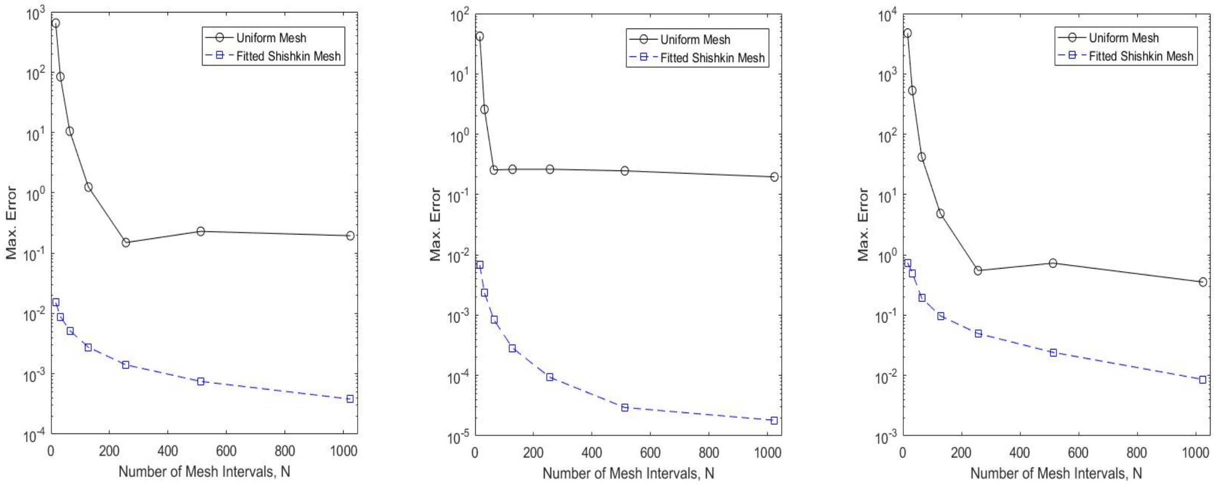

Figure 3.1.4: Comparison of ϵ-uniform maximum point-wise error from $N = 16$ to $N = 1024$ of: (a) Example 1 when $p = 2$, (b) Example 2 when $p = 3$ and (c) Example 3 when $p = 2$.

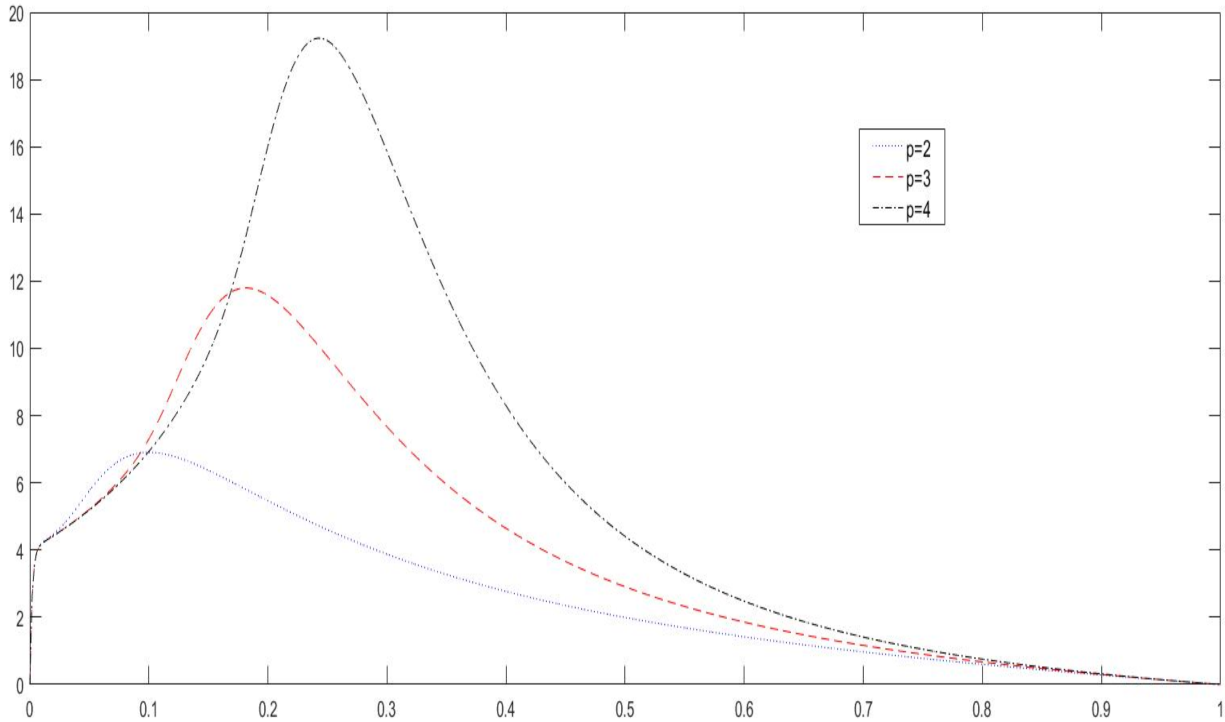

Figure 3.1.5: Fitted numerical solutions of Example 3 for $\epsilon = 2^{-20}$, $N = 512$, with different values of p.

3.2 Summary of Results

Numerical errors $E_{\infty}^{N,\epsilon}$ presented in Tables 3.1.2 and 3.1.5 for various values of the parameter ϵ, p and collocation nodes implies that the proposed method with fitted Shishkin mesh shows a greater agreement with the exact solution as the mesh size is refined. To further illustrate the applicability of the proposed method numerical solution profiles have been plotted in Fig. 3.1.1 and Fig. 3.1.2 for Examples 1 and 2 for the exact solution versus computed solutions obtained for the different values of ϵ and N on uniform mesh and Shishkin mesh. On a uniform mesh, the numerical method fails to capture the singularly perturbed oscillatory nature of the solution as the perturbation parameter $\epsilon \to 0$. It has been seen that the exact and the numerical solution are identical in most of the regions of the domain, except in the boundary layer region near $x = 0$. To control these deviations in the boundary layer region, we use piecewise uniform Shishkin mesh which is finely concentrated in the boundary layer region and the resulting behaviour can be seen in the figures. From the numerical solution profiles given in Fig. 3.1.5, we observe that with increasing values of p, the hump gets thin and shifts away from the boundary layer region. The plots displayed in Fig. 3.1.4 clearly demonstrate the higher accuracy of the proposed numerical scheme on a fitted Shishkin mesh as compared to the uniform mesh. Comparative results obtained by the collocation method are presented along with those obtained by other researchers. Table 3.1.3 show the proposed scheme is more accurate and efficient than the classical finite-difference scheme.

4 Conclusion

A numerical method is presented to solve a linear second-order singularly perturbed multiple boundary turning point problems. In general, the numerical treatment of turning point problems is much more complicated than singular perturbation problems without turning points. B-splines are used because they yield results of higher accuracy as compared with those of polynomial interpolation. The most significant virtue of the collocation method is its ease in computation and application. Collocation with B-splines leads to banded matrices as opposed to full matrices using polynomials, trigonometric functions and other well-known non-piecewise approximates. These methods can be closely related to Galerkin methods, hence to finite-element methods (FEM). In comparison to FEM, the collocation matrix involves no integration or numerical quadrature, for complicated coefficients of differential equations, hence reduces the bandwidth and operation count. For the cubic B-spline collocation method, the number of nonzero terms in a row of the coefficient matrix is equal to the number of nonzero basis functions at the corresponding mesh point, that is, 3, and hence the bandwidth is 1. However, in the case of the cubic B-spline finite-element Galerkin method, the products of cubic B-spline basis functions are integrated to compute the elements of the defining matrix which gives the bandwidth of 3. Thus, the collocation method has less order computational complexity as compared to the Galerkin method.

Solutions of such problems at small values of ϵ have a abrupt behaviour in the neighborhood of the boundary layer. It has been seen that the exact and numerical solutions without fitted mesh are identical for most of the domain except in the boundary layer regions. To control these deviations in the boundary layer region, piecewise uniform S-mesh is used that gives improved accuracy as compared to related uniform mesh scheme. A brief analysis has been carried out and the method is shown to be ϵ-uniformly convergent. It is seen from Tables 3.1.2, 3.1.5 and 3.1.7 that for a fixed value of ϵ, the values of maximum pointwise errors $E_{\infty}^{N,\epsilon}$ decreases as the number of mesh points N increases, which shows that the convergence is uniform concerning mesh parameter. The performance of the proposed scheme is investigated by comparing the results for some well-known problems and observed that the accuracy in the numerical results is comparable and better to those by existing methods. Thus, the method works nicely for small values of ϵ and numerical results support theoretical predictions and exhibit good physical behaviour.

Future Research Directions

Singularly perturbed turning point(s) problems were first studied in the late 1960s. The analysis of singularly perturbed problems had previously been restricted to non-vanishing coefficients. This section highlights the research perspectives resulting from this thesis. In the previous chapter, we constructed a cubic B-spline collocation scheme to numerically solve the singularly perturbed multiple turning point problem. In addition, it would be interesting to consider higher-order quintic B-splines on Shishkin mesh and Bakhvalov mesh to further improve the results in comparison with cubic B-splines. To add to the subject, a few extensive problems may be considered:

A class of one-dimensional singularly perturbed problem with a boundary turning point

$$-\epsilon u'' - b(x)u' = f(x), \quad u(0) = u(1) = 0, \quad b(x) > 0, \quad x \in (0,1).$$

where $b(x) = x^p$, $p > 0$. This type of problem was previously studied by Chen et al. [22] where they provided an error estimate for a broad class of layer-adapted grids with a priori or a posteriori information on second-order derivatives and can also be applied to other problems.

A class of singularly perturbed time-dependent convection-diffusion problems with a boundary turning point on a rectangular domain

$$\begin{aligned}
&L_\epsilon u(x,t) \equiv (\epsilon u_{xx} + a u_x - b u_t - du)(x,t) = f(x,t) \ \text{ in } D = \Omega \times (0,t), \quad \Omega = (0,1)\\
&u(x,t) = g(x,t) \ \text{ on } \Gamma,\\
&a(x,t) = a_0(x,t)x^p, \quad p \geq 1 \ \ \forall (x,t) \in \bar{D}, \quad a_0(x,t) \geq \alpha > 0,\\
&b(x,t) \geq \beta > 0, \quad d(x,t) \geq \delta \geq 0 \ \ \forall (x,t) \in \bar{D},\\
&\Gamma = \bar{D} = \Gamma_l \cup \Gamma_b \cup \Gamma_r, \quad \Gamma_l = \{(0,t) | 0 \leq t \leq T\},\\
&\Gamma_b = \{(x,0) | 0 \leq x \leq 1\}, \quad \Gamma_r = \{(1,t) | 0 \leq t \leq T\}.
\end{aligned}$$

Here a_0, b, d, f and g are sufficiently regular and f, g at the corners of the domain satisfy sufficient compatibility conditions. Dunne et al. [28] investigated this problem on a layer-adapted piecewise-uniform Shishkin mesh using the backward finite-difference method on a time derivative.

A system of singularly perturbed nonlinear differential equations with attractive turning point

$$\begin{aligned}
&-\epsilon u''(x) + (xu(x))' + a(x)u(x) = f(x), \quad -1 < x < 1, \quad u(-1) = A, \quad u(1) = B,\\
&-\epsilon y''(x) + (xy(x))' + b(x, y(x)) = 0, \quad -1 < x < 1, \quad y(-1) = A, \quad y(1) = B,
\end{aligned}$$

where $a(x) \geq 1$, $a(0) \geq 1 + \delta$, $b_y(x,y) \geq 1 + \delta$ and $\delta > 0$. Clavero and Lisbona [23] studied and gave numerical results using exponentially fitted schemes for the integration of linear problem and Samarskii's scheme of semilinear problem.

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